NONEXTENSIVE STATISTICAL MECHANICS OF COMPLEX SYSTEMS

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J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

C. Scribner's Sons, New York, 1902; Yale University Press, New Haven, (1981), page 35

In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] *to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space […]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. […]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

POSTULATED ENTROPIC FUNCTIONALS

	$p_i = \frac{1}{W} \quad (\forall i)$ **equiprobability**	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG entropy (*q* =*1*)	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* (*q real*)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

additive

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

Conformally invariant geometry

nonadditive (if $q \neq 1$)

Possible generalization of Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]

DEFINITIONS: *q-logarithm*: $\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0;\ \ln_1 x = \ln x)$

q-exponential: $e_q^x \equiv [1+(1-q)\,x]^{\frac{1}{1-q}} \quad (e_1^x = e^x)$

Hence, the entropies can be rewritten:

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k\ \ln W$	$k\sum_{i=1}^{W} p_i \ln\frac{1}{p_i}$
entropy S_q $(q\in R)$	$k\ \ln_q W$	$k\sum_{i=1}^{W} p_i \ln_q\frac{1}{p_i}$

TYPICAL SIMPLE SYSTEMS:

$$\text{e.g.,}\quad W(N) \propto \mu^N \quad (\mu > 1)$$

Short-range space-time correlations

Markovian processes (short memory), Additive noise

Strong chaos (positive maximal Lyapunov exponent), **Ergodic**, Riemannian geometry

Short-range many-body interactions, weakly quantum-entangled subsystems

Linear/homogeneous Fokker-Planck equations, Gausssians

→ Boltzmann-Gibbs entropy (additive)

→ Exponential dependences (Boltzmann-Gibbs weight, ...)

TYPICAL COMPLEX SYSTEMS:

$$\text{e.g.,}\quad W(N) \propto N^{\rho} \quad (\rho > 0)$$

Long-range space-time correlations

Non-Markovian processes (long memory), Additive and multiplicative noises

Weak chaos (zero maximal Lyapunov exponent), **Nonergodic**, Multifractal geometry

Long-range many-body interactions, strongly quantum-entangled sybsystems

Nonlinear/inhomogeneous Fokker-Planck equations, *q*-Gaussians

→ Entropy S*q* (nonadditive)

→ *q*-exponential dependences (asymptotic power-laws)

ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for any two probabilistically independent systems A and B,

$$S(A+B) = S(A) + S(B)$$

Therefore, since

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)\, S_q(A)\, S_q(B) \;,$$

S_{BG} and $S_q^{Renyi}\,(\forall q)$ are additive, and S_q $(\forall q \neq 1)$ is nonadditive .

EXTENSIVITY:

Consider a system $\Sigma \equiv A_1 + A_2 + \ldots + A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N.

An entropy is extensive if

$$0 < \lim_{N\to\infty} \frac{S(N)}{N} < \infty \;,\; \textit{i.e.,}\;\; S(N) \propto N \quad (N \to \infty)$$

PHYSICAL REVIEW E **78**, 021102 (2008)

Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

$$\hat{\mathcal{H}} = -\sum_{j=1}^{N-1} \left[(1+\gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1-\gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z\right]$$

$|\gamma| = 1 \quad \rightarrow \quad$ *Ising ferromagnet*

$0 < |\gamma| < 1 \quad \rightarrow \quad$ *anisotropic XY ferromagnet*

$\gamma = 0 \quad \rightarrow \quad$ *isotropic XY ferromagnet*

$\lambda \equiv$ *transverse magnetic field*

$L \equiv$ *length of a block within a* $N \rightarrow \infty$ *chain*

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

$\rho_N \equiv$ ground state $(T = 0)$ of the N-system

(assuming $\lambda^{xy} = +0$)

$\Rightarrow \rho_N^2 = \rho_N \Rightarrow Tr\rho_N^2 = 1$

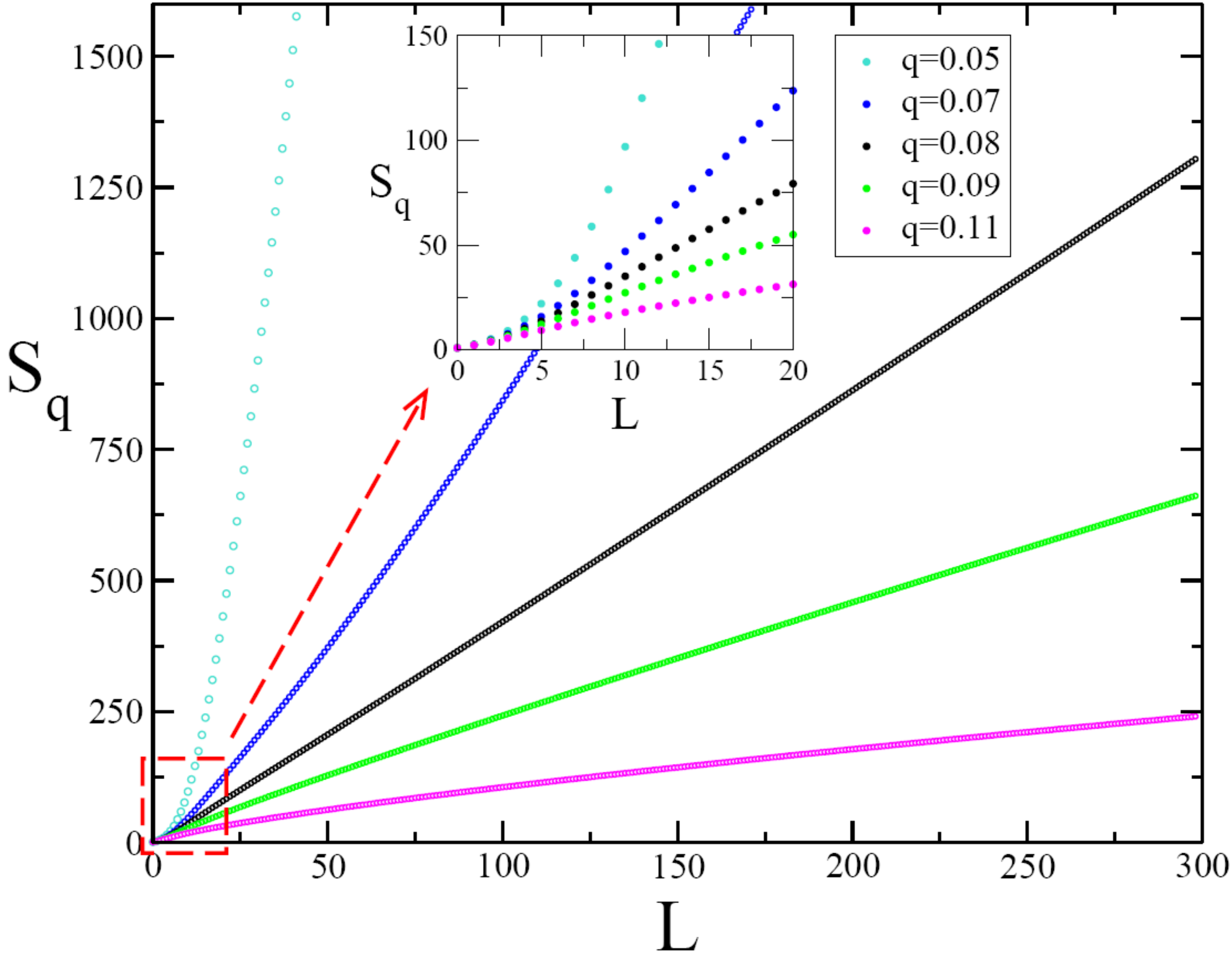
$\Rightarrow \rho_N$ is a pure state

$\Rightarrow S_q(N) = 0 \quad (\forall q,\ \forall N)$

In contrast, $\rho_L \equiv Tr_{N-L}\ \rho_N$ satisfies $Tr\rho_L^2 < 1$

$\Rightarrow \rho_L$ is a mixed state

$\Rightarrow S_q(N, L) > 0$



F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

Using a Quantum Field Theory result

*in P. Calabrese and J. Cardy, JSTAT P*06002 (2004)

we obtain, at the critical transverse magnetic field,

$$q_{ent} = \frac{\sqrt{9+c^2}-3}{c}$$

with $c \equiv$ central charge in conformal field theory

Hence

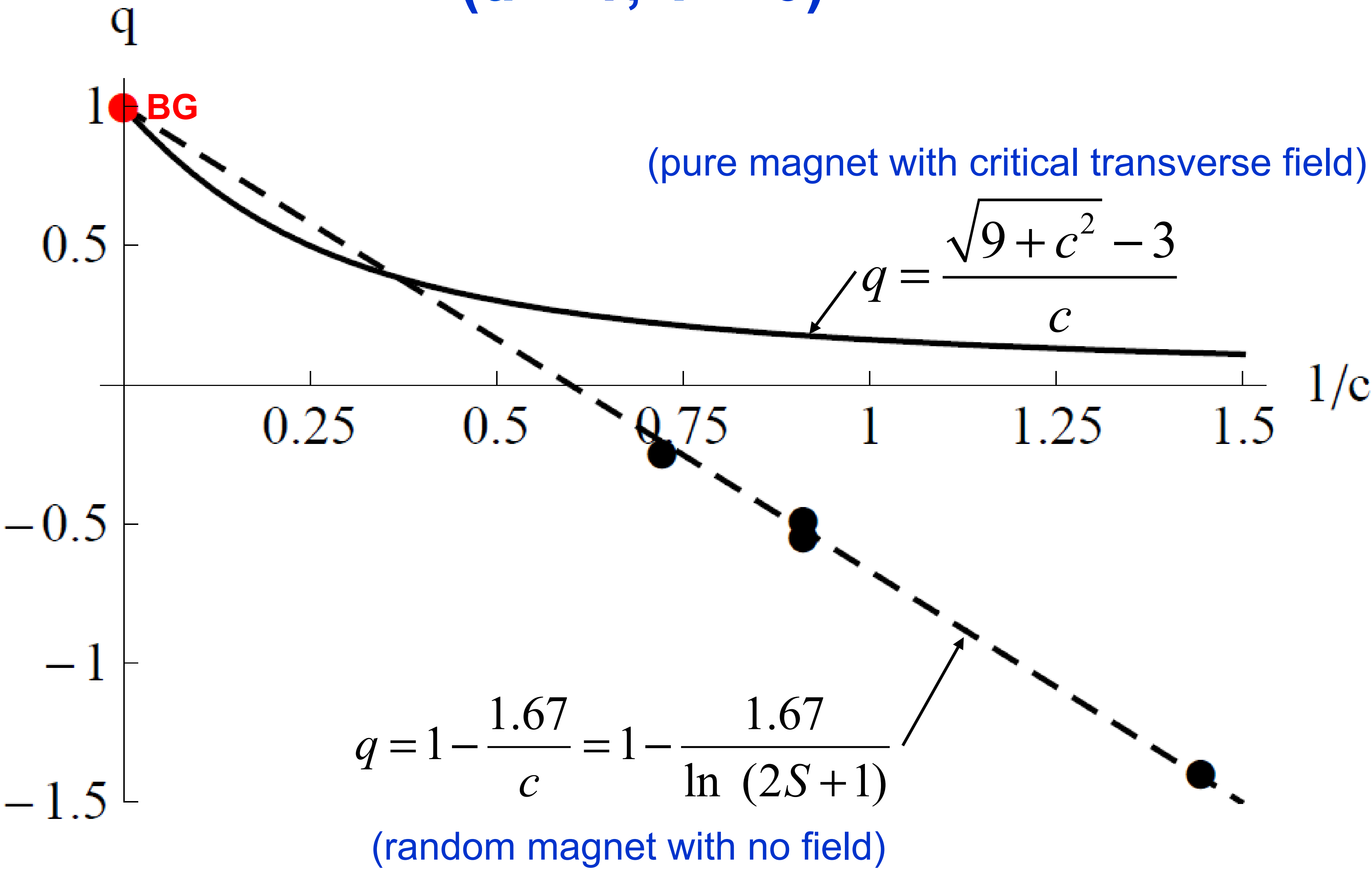
Ising and anisotropic XY ferromagnets $\Rightarrow \; c = \frac{1}{2} \;\Rightarrow\; q_{ent} = \sqrt{37}-6 \;\approx 0.0828$

and

Isotropic XY ferromagnet $\Rightarrow \; c = 1 \;\Rightarrow\; q_{ent} = \sqrt{10}-3 \;\approx 0.1623$

F. Caruso and C. T., Phys Rev E **78**, 021101 (2008)

$(d = 1;\ T = 0)$



A Saguia and MS Sarandy, Phys Lett A **374**, 3384 (2010)

SYSTEMS	ENTROPY S_{BG} (additive)	ENTROPY S_q (q<1) (nonadditive)
Short-range interactions, weakly entangled blocks, etc	**EXTENSIVE**	**NONEXTENSIVE**
Long-range interactions (QSS), strongly entangled blocks, etc	**NONEXTENSIVE**	**EXTENSIVE**

quarks-gluons, plasma, curved space ...?

When entropy does not seem extensive

Earlier speculations about the entropy of black holes has prompted an ingenious calculation suggesting that entropy may (in special circumstances) be the same inside and outside an arbitrary boundary.

Everybody who knows about entropy knows that it is an extensive property, like mass or enthalpy. That, of course, is why the entropy of some substance will be quoted as so much per gram, or mole. If you then take two grams, or two moles, of the same material under the same conditions, the entropy will be twice as much. And there should be no confusion about the units; the simple Carnot definition of a change of entropy in a reversible process is the heat transfer divided by the absolute temperature, so that the units of entropy are simply those of energy divided by temperature, joules per degree (kelvin) in the SI system. The definitions of the Gibbs and Helmholtz free energies would be dimensionally discordant for that reason were it not that entropy (S) always turns up multiplied by temperature T. So much will readily be agreed.

Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is *disorder* measured? By the number of ways in which the constituents of some material (the atoms and molecules) can be rearranged without changing its properties and without energetic consequences. But now there comes a snag.

Like any extensive property, the combined entropy of two separate chunks of material should be the *sum* of the two entropies, but the number of rearrangements of the combined system must be the *product* of the numbers of ways in which the two parts separately can be rearranged. How to reconcile that with extensivity? By supposing entropy is proportional not to the number of rearrangements (technically called "complexions"), but with the logarithm thereof. And because entropy decreases as disorder increases, the constant of proportionality must be a negative (real) number.

From that it follows that $S = S_0 - K\log N$, where K is a positive constant with the dimensions of entropy, N is a number (without dimensions) measuring disorder and S_0 is an arbitrary constant entropy. All that is simply a precis of the standard introductory chapter in statistical mechanics textbooks, most of which go on to show how to calculate the properties of assemblages of, say, diatomic molecules from a knowledge of their individual behaviour. Because the number of complexions of a particular state of an assemblage is invariably a function of the number (n) of molecules it contains, usually in the form of $n!$, because n is usually large and because $\log(n!)$ can then be approximated by $n\log n$, the extensive property of entropy then follows simply from the appearance of the leading factor n: entropy is proportional to the number of molecules.

That is what the textbooks say. It also makes sense of what is known of the thermodynamics of the real world. In a sample of a diatomic gas, for example, there are vibrations (one) and rotations (two) as well as three rectilinear degrees of freedom. But the problem is to tell how the energy available is distributed among the different degrees of freedom. The arithmetic simplifies marvellously because (in this case) each molecule and each of its degrees of freedom is independent. The best measure of disorder works out at $N = Z^n$, where n is the number of molecules, and where Z, which must be a

…

A bit of quantum mechanics goes into the argument as well, notably the notion of the density matrix — an artificially constructed operator (on quantum states) that is well suited to the discussion of systems in which one part (say the black hole) is singled out for attention while the remainder (the Universe outside it) is dealt with in less detail, perhaps because some averaging process is appropriate, or because the whole problem may not be calculable at all. (In Dirac's notation, the density matrix corresponding to some state of the whole Universe would be represented as $|1\rangle\langle 1|$, where "1" is simply the name for a particular state of the Universe.) What matters, where entropy is concerned, is that the density matrix, like all matrices, has eigenvalues from which the entropy can be calculated.

So imagine that the Universe is partitioned into two parts by means of a closed boundary of some kind and filled with a

…

dealt with explicitly, as other entropy calculations are made. And that could be exceedingly important.

John Maddox

Jacob D. Bekenstein
Stephen W. Hawking
Gerard 't Hooft
Leonard Susskind
Stephen Lloyd
Juan M. Maldacena
…

When entropy does not seem extensive
John Maddox, Nature 365, 103 (1993)

Everybody who knows about entropy knows that it is an extensive property, like mass or enthalpy. [...] Of course, there is more than that to entropy, which is also a measure of disorder. Everybody also agrees on that. But how is disorder measured? [...] So why is the entropy of a black hole proportional to the square of its radius, and not to the cube of it? To its surface area rather than to its volume?

PHYSICAL REVIEW D VOLUME 13, NUMBER 2 15 JANUARY 1976

Black holes and thermodynamics*

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(Received 30 June 1975)

A black hole of given mass, angular momentum, and charge can have a large number of different unobservable internal configurations which reflect the possible different initial configurations of the matter which collapsed to produce the hole. The logarithm of this number can be regarded as the entropy of the black hole and is a measure of the amount of information about the initial state which was lost in the formation of the black hole. If one makes the hypothesis that the entropy is finite, one can deduce that the black holes must emit thermal radiation at some nonzero temperature. Conversely, the recently derived quantum-mechanical result that black holes do emit thermal radiation at temperature $\kappa\hbar/2\pi kc$, where κ is the surface gravity, enables one to prove that the entropy is finite and is equal to $c^3A/4G\hbar$, where A is the surface area of the event horizon or boundary of the black hole. Because black holes have negative specific heat, they cannot be in stable thermal equilibrium except when the additional energy available is less than 1/4 the mass of the black hole. This means that the standard statistical-mechanical canonical ensemble cannot be applied when gravitational interactions are important. Black holes behave in a completely random and time-symmetric way and are indistinguishable, for an external observer, from white holes. The irreversibility that appears in the classical limit is merely a statistical effect.

ENTROPIES

$$S_{BG} = k_B \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$$ $\rightarrow$ *additive*

$$S_q = k_B \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} \qquad (S_1 = S_{BG})$$ $\rightarrow$ *nonadditive* if $q \neq 1$ C. T. (1988)

$$S_\delta = k_B \sum_{i=1}^{W} p_i \left(\ln \frac{1}{p_i} \right)^\delta \qquad (S_1 = S_{BG})$$ $\rightarrow$ *nonadditive* if $\delta \neq 1$ C. T. (2009)

$$S_{q,\delta} = k_B \sum_{i=1}^{W} p_i \left(\ln_q \frac{1}{p_i} \right)^\delta \qquad (S_{q,1} = S_q;\ S_{1,\delta} = S_\delta;\ S_{1,1} = S_{BG})$$

$\rightarrow$ *nonadditive* if $(q,\delta) \neq (1,1)$

C. T. and L.J.L. Cirto (2011)

1202.2154 [cond-mat.stat-mech]

EXTENSIVITY OF THE ENTROPY $(N \to \infty)$

If $W(N) \sim \mu^N \quad (\mu > 1)$

$$\Rightarrow \; S_{BG}(N) = k_B \ln W(N) \; \propto N$$

If $W(N) \sim N^\rho \quad (\rho > 0)$

$$\Rightarrow \; S_q(N) \; = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}$$

$$\Rightarrow \; S_{q=1-1/\rho}(N) \; \propto N$$

If $W(N) \sim \nu^{N^\gamma} \quad (\nu > 1;\; 0 < \gamma < 1)$

$$\Rightarrow \; S_\delta(N) = k_B \left[\ln W(N)\right]^\delta \propto N^{\gamma\,\delta}$$

$$\Rightarrow \; S_{\delta=1/\gamma}(N) \; \propto N$$

Hawking, string theory, etc, yield

$$S_{BG}^{black\ hole}(N) \equiv k_B \ln W(N) \propto L^2 \propto N^{2/3} \qquad (N \propto L^3)$$

More generally, we have

$$S_{BG}(N) = k_B \ln W(N) \propto L^{d-1} \propto N^{\frac{d-1}{d}} \qquad (d > 1)$$

hence

$$W(N) \propto \Phi(N)\ \nu^{N^{\frac{d-1}{d}}} \qquad \left(\text{with } \lim_{N\to\infty} \frac{\ln \Phi(\mathrm{N})}{N^{\frac{d-1}{d}}} = 0 \right)$$

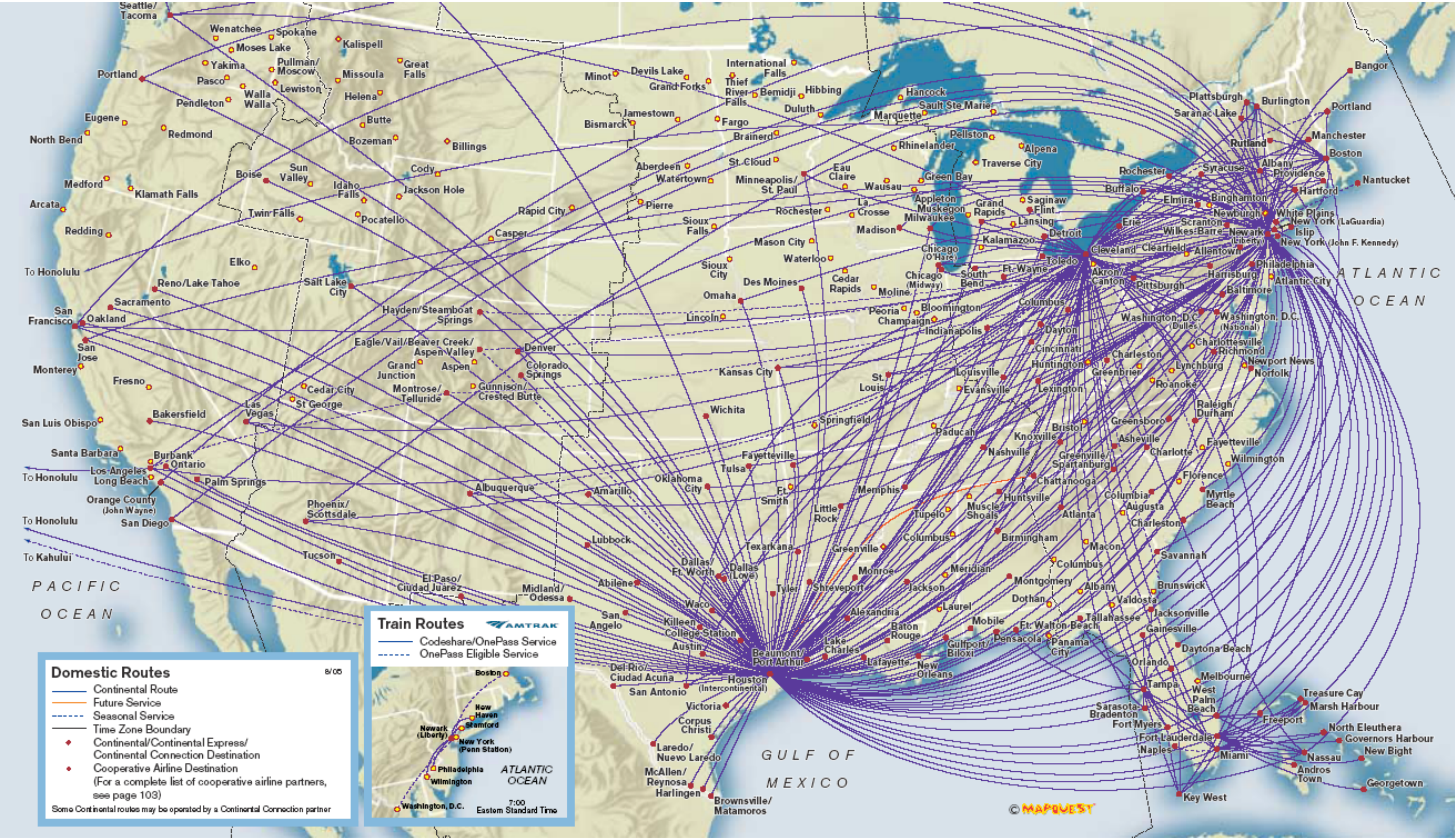
hence the entropy which is extensive is S_δ with $\delta = \dfrac{d}{d-1}$

i.e., $$S_\delta(N) = k_B \sum_{i=1}^{W(N)} p_i \left(\ln \frac{1}{p_i} \right)^{\frac{d}{d-1}} \qquad (d > 1)$$

Consequently $$S_{\delta=3/2}^{black\ hole}(N) = k_B \sum_{i=1}^{W(N)} p_i \left(\ln \frac{1}{p_i} \right)^{\frac{3}{2}} \propto N \propto L^3 \qquad !!!$$

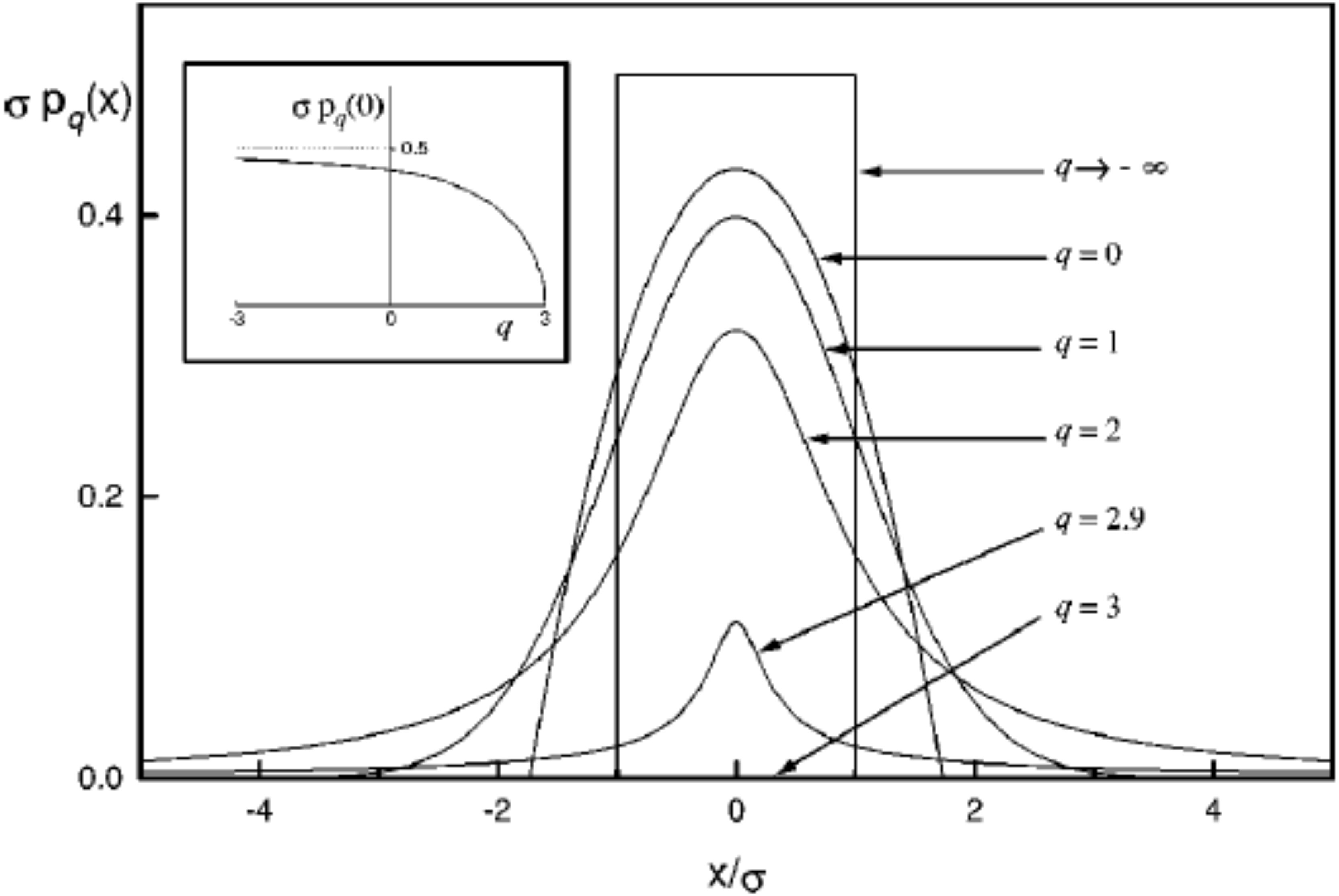
SYSTEMS $W(N)$	ENTROPY S_{BG} (ADDITIVE)	ENTROPY S_q $(q \neq 1)$ (NONADDITIVE)	ENTROPY S_δ $(\delta \neq 1)$ (NONADDITIVE)
$\sim \mu^N$ $(\mu > 1)$	EXTENSIVE	NONEXTENSIVE	NONEXTENSIVE
$\sim N^\rho$ $(\rho > 0)$	NONEXTENSIVE	EXTENSIVE	NONEXTENSIVE
$\sim \nu^{N^\gamma}$ $(\nu > 1;$ $0 < \gamma < 1)$	NONEXTENSIVE	NONEXTENSIVE	EXTENSIVE

King Thutmosis I
18^{th} Dynasty
circa 1500 BC



Continental Airlines

q-GAUSSIANS: $p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \dfrac{1}{\left[1+(q-1)\,(x/\sigma)^2\right]^{\frac{1}{q-1}}}$ $(q<3)$



D. Prato and C. T., Phys Rev E **60**, 2398 (1999)

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

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See also:

H.J. Hilhorst, JSTAT P10023 (2010)

M. Jauregui and C. T., Phys Lett A **375**, 2085 (2011)

M. Jauregui, C. T. and E.M.F. Curado, JSTAT P10016 (2011)

A. Plastino and M.C. Rocca, Physica A and Milan J Math (2012)

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ *-scaled attractor* $\mathrm{F}(x)$ *when summing* $N\to\infty$ q*-independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2[f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q\equiv 2q-1,\ q_1=\frac{1+q}{3-q} \right)$

	$q=1$ [independent]	$q\neq 1$ (i.e., $Q\equiv 2q-1\neq 1$) [globally correlated]
$\sigma_Q<\infty$ $(\alpha=2)$	$\mathrm{F}(x)=$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathrm{F}(x)=G_q(x)\equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x)\sim\begin{cases} G(x) & \text{if } \lvert x\rvert \ll x_c(q,2) \\ f(x)\sim C_q/\lvert x\rvert^{2/(q-1)} & \text{if } \lvert x\rvert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q\to 1} x_c(q,2)=\infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q\to\infty$ $(0<\alpha<2)$	$\mathrm{F}(x)=$ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x\rvert\to\infty$ *behavior* $\mathrm{L}_\alpha(x)\sim\begin{cases} G(x) \\ \quad \text{if } \lvert x\rvert \ll x_c(1,\alpha) \\ f(x)\sim C_\alpha/\lvert x\rvert^{1+\alpha} \\ \quad \text{if } \lvert x\rvert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha\to 2} x_c(1,\alpha)=\infty$ Levy-Gnedenko CLT	$\mathrm{F}(x)=L_{q,\alpha}$, *with same* $\lvert x\rvert\to\infty$ *asymptotic behavior* $L_{q,\alpha}\sim\begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\ \alpha}(x)\sim C^*_{q,\alpha}/\lvert x\rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \qquad \textit{(intermediate regime)} \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\ 2}(x)\sim C^L_{q,\alpha}/\lvert x\rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \qquad \textit{(distant regime)} \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

PHYSICAL REVIEW E **00**, 001100 (2011)

Group entropies, correlation laws, and zeta functions

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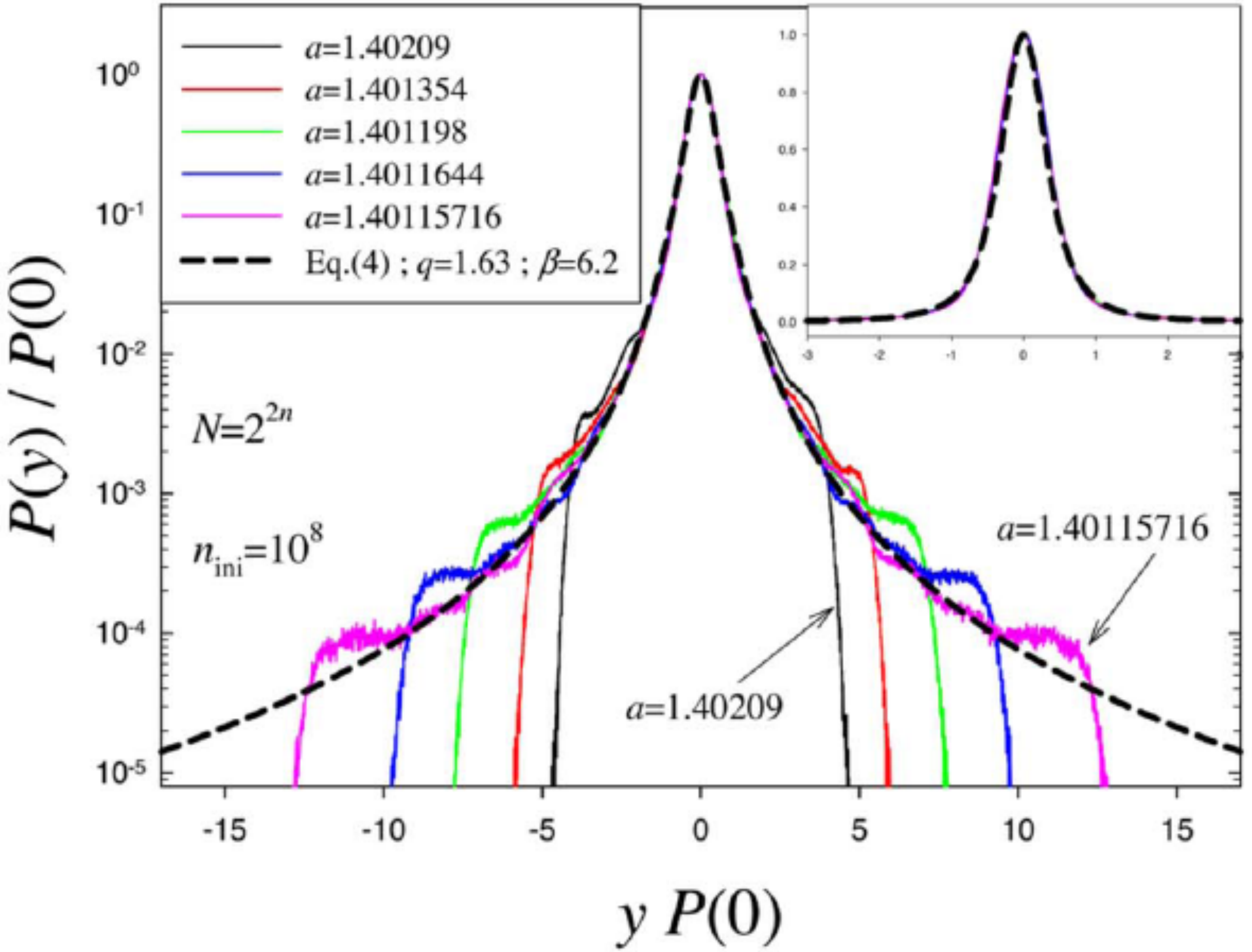
The notion of group entropy is proposed. It enables the unification and generaliztion of many different definitions of entropy known in the literature, such as those of Boltzmann-Gibbs, Tsallis, Abe, and Kaniadakis. Other entropic functionals are presented, related to nontrivial correlation laws characterizing universality classes of systems out of equilibrium when the dynamics is weakly chaotic. The associated thermostatistics are discussed. The mathematical structure underlying our construction is that of formal group theory, which provides the general structure of the correlations among particles and dictates the associated entropic functionals. As an example of application, the role of group entropies in information theory is illustrated and generalizations of the Kullback-Leibler divergence are proposed. A new connection between statistical mechanics and zeta functions is established. In particular, Tsallis entropy is related to the classical Riemann zeta function.

$$S_q \leftrightarrow \varsigma(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \mathrm{L}$$

LOGISTIC MAP AT THE EDGE OF CHAOS:



$q = 1.63$

$\beta = 6.2$

U. Tirnakli, C. T. and C. Beck, Phys Rev E **79** (2009) 056209

CONSERVATIVE MC MILLAN MAP:

$$x_{n+1} = y_n$$

$$y_{n+1} = -x_n + 2\mu \frac{y_n}{1+y_n^2} + \varepsilon y_n$$

$\mu \neq 0 \Leftrightarrow$ nonlinear dynamics

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press

$(\mu, \varepsilon) = (1.6, 1.2)$ $(\lambda_{\max} \approx 0.05)$

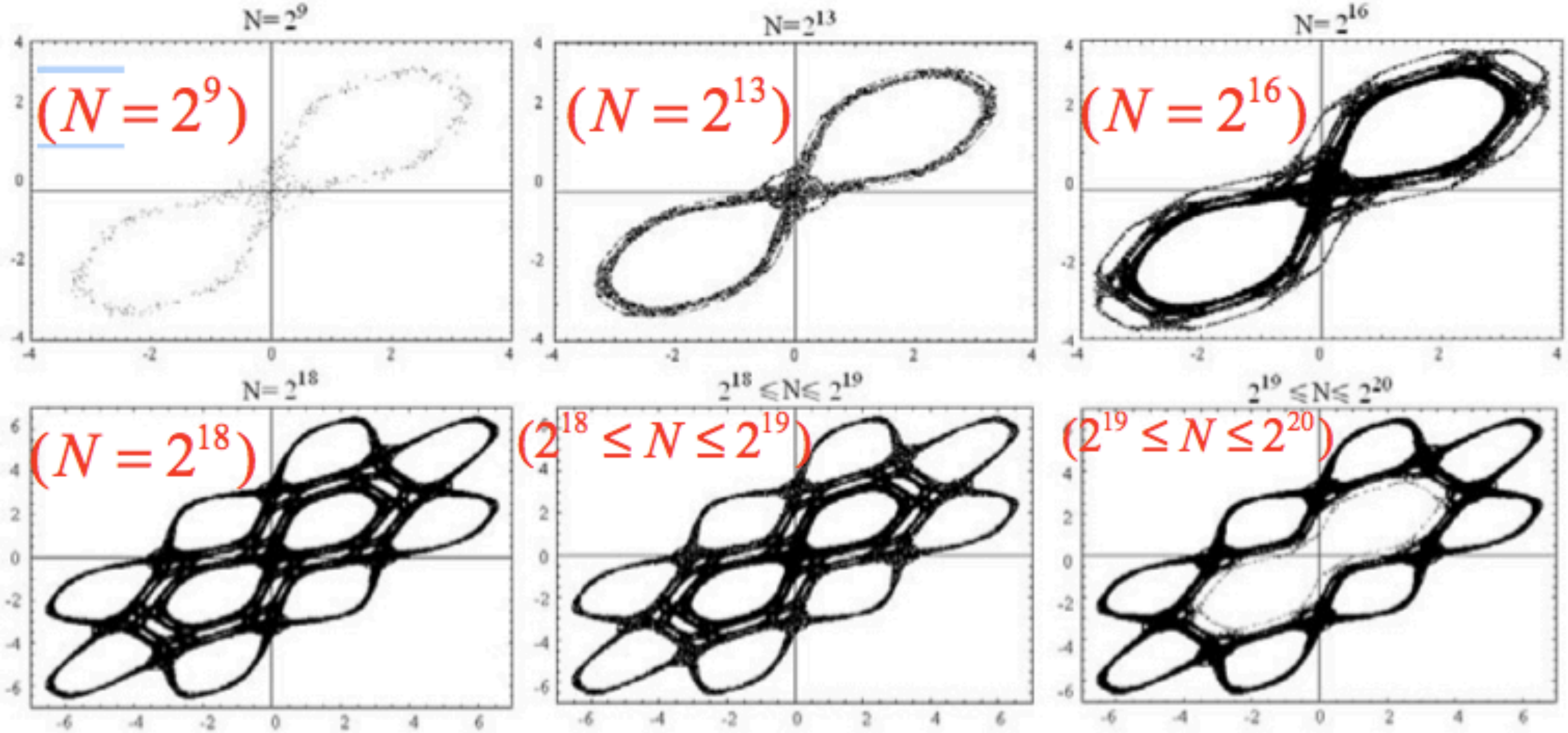
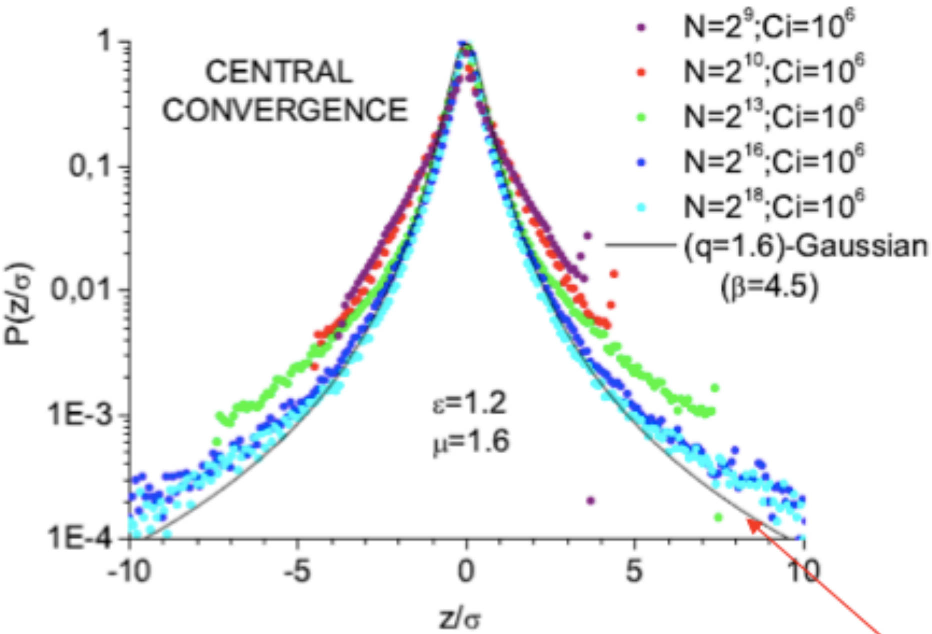
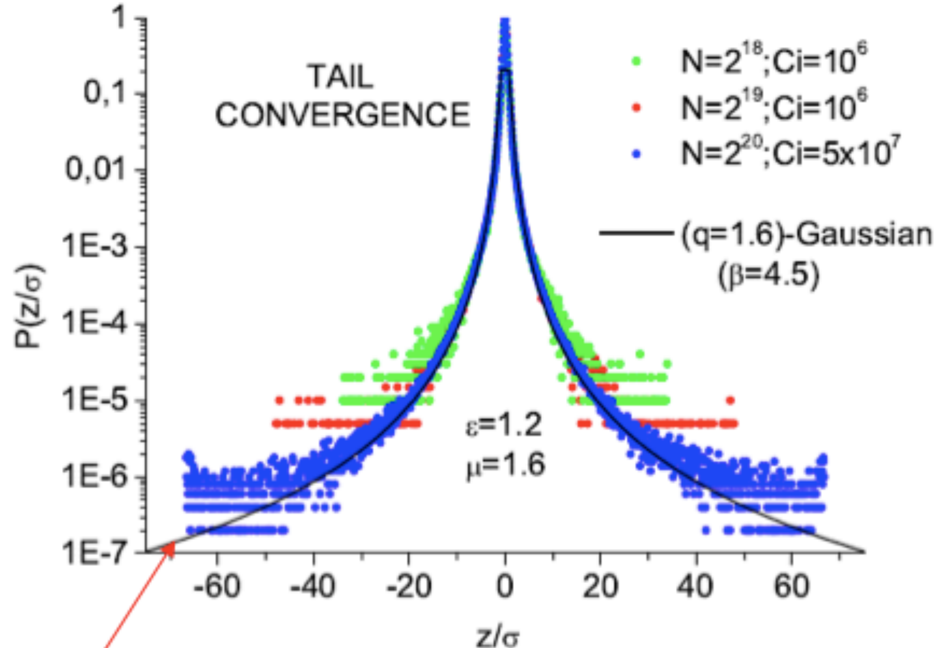


FIG. 10. Structure of phase space plot of Mc. Millan perturbed map for parameter values $\mu = 1.6$ and $\epsilon = 1.2$, starting form a randomly chosen initial condition in a square $(0, 10^{-6}) \times (0, 10^{-6})$, and for $i = 1 \ldots N$ $(N = 2^{10}, 2^{13}, N^{16}, N^{18})$ iterates.

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press





$$p \propto e_q^{-\beta (z/\sigma)^2}$$

with $(q, \beta) = (1.6, 4.5)$

G. Ruiz, T. Bountis and C. T.
Int J Bifurcat Chaos (2012), in press

CORRELATIONS IN COUPLED LOGISTIC MAPS AT THE EDGE OF CHAOS IN THE PRESENCE OF GLOBAL NOISE

We consider a linear chain of N coupled maps with periodic boundary conditions in a noisy environment:

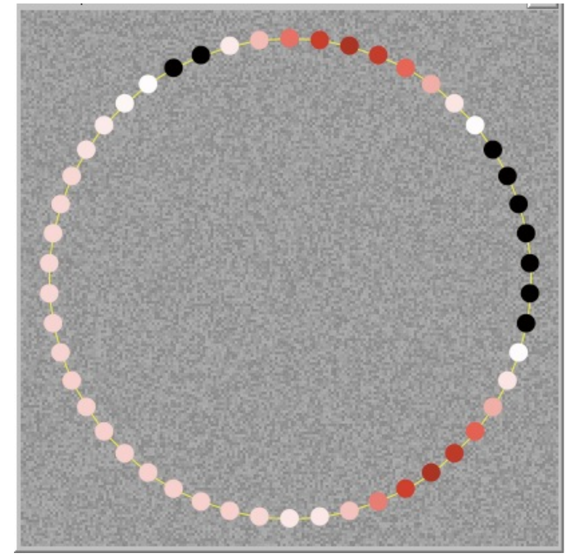
$$x_{t+1}^i = (1-\varepsilon) f(x_t^i) + \frac{\varepsilon}{2}[f(x_t^{i-1}) + f(x_t^{i+1})] + \sigma_t$$

$\boxed{\sigma_t \in [0, \sigma_{\max}]}$ additive noise

with $\varepsilon \in [0,1]$ coupling strength

and $f(x_t^i) = 1 - \mu (x_t^i)^2 \quad \mu \in [0,2]$ i-th logistic map ($i = 1 \ldots N$)

[zero noise: N.B. Ouchi and K. Kaneko, Chaos **10**, 359 (2000)]



$\boxed{\text{edge of chaos: } \mu = 1.4011551}$

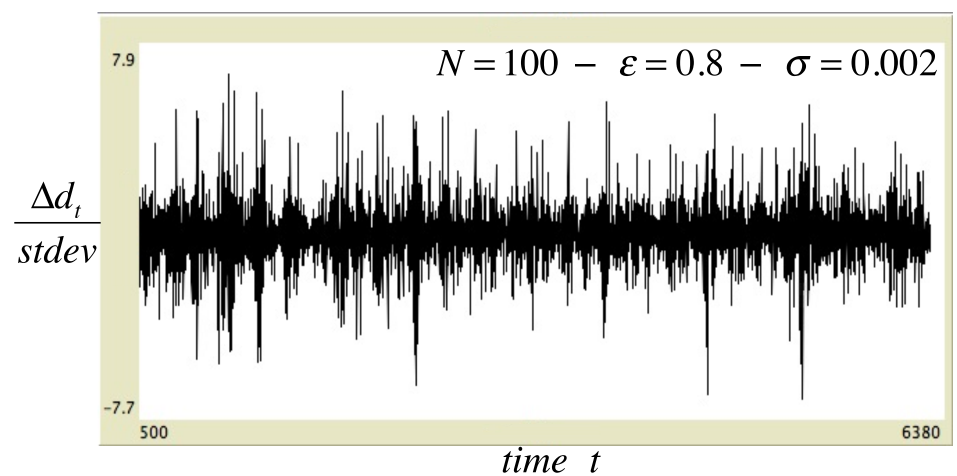
global parameter

$$d_t = \Sigma_{i=1}^N |x_t^i - < x_t^i > |$$

time returns

$$\Delta d_t = d_{t+\tau} - d_t$$

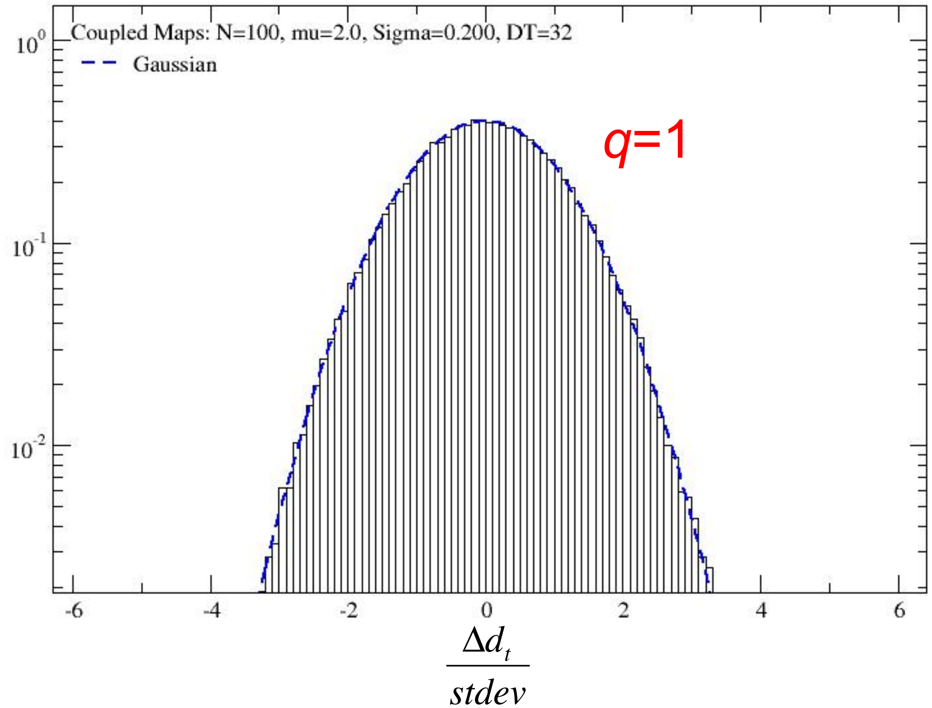
Intermittency in the normalized returns time-series



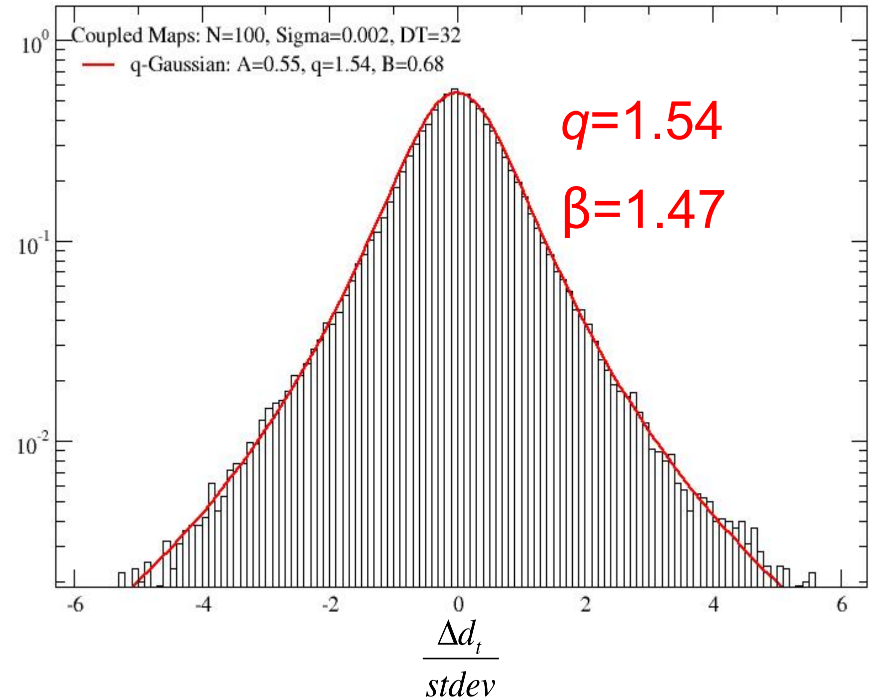
A. Pluchino, A. Rapisarda and C. T. (2012), 1206.2152 [cond-mat.stat-mech]

$$N = 100;\ \varepsilon = 0.8;\ \sigma_{\max} = 0.002;\ \tau = 32$$

Chaotic Regime: $\mu = 2.0$



Edge of chaos: $\mu = 1.4011551$



A. Pluchino, A. Rapisarda and C. T. (2012), 1206.2152 [cond-mat.stat-mech]

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Thermostatistics in the neighbourhood of the π-mode solution for the Fermi–Pasta–Ulam β system: from weak to strong chaos

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Online at stacks.iop.org/JSTAT/2010/P04021
doi:10.1088/1742-5468/2010/04/P04021

Abstract. We consider a π-mode solution of the Fermi–Pasta–Ulam β system. By perturbing it, we study the system as a function of the energy density from a regime where the solution is stable to a regime where it is unstable, first weakly and then strongly chaotic. We introduce, as an indicator of stochasticity, the ratio ρ (when it is defined) between the second and the first moment of a given probability distribution. We will show numerically that the transition between weak and strong chaos can be interpreted as the symmetry breaking of a set of suitable dynamical variables. Moreover, we show that in the region of weak chaos there is numerical evidence that the thermostatistic is governed by the Tsallis distribution.

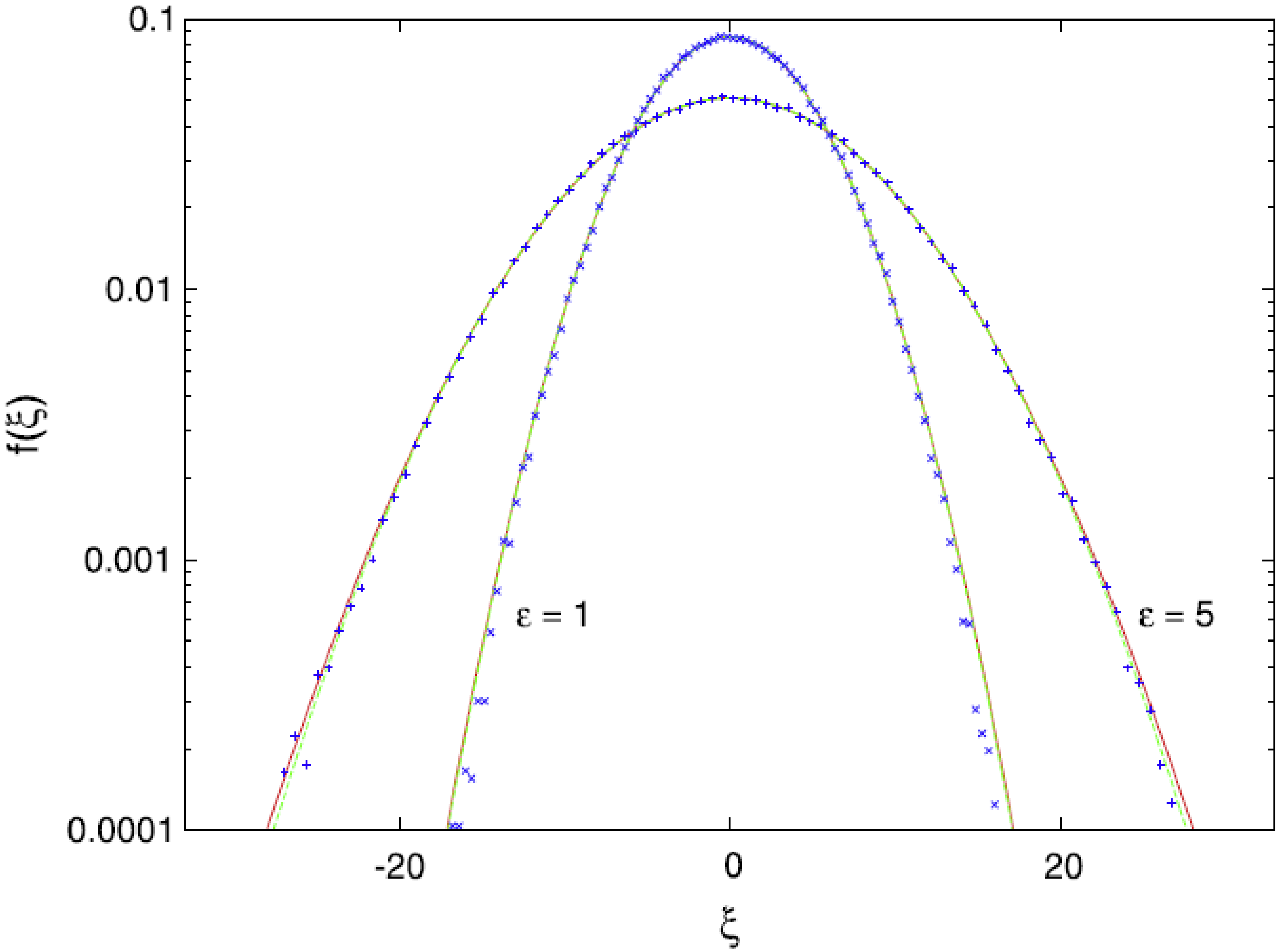

Figure 5. Plot on a linear-log scale of the numerical distribution $f(\xi)$ (blue points) fitted with a Tsallis distribution (red) and a Gauss distribution (green) for $N = 128$, $\epsilon = 1$ and 5. In both cases the Tsallis and Gaussian distributions essentially overlap.

M. Leo, R.A. Leo and P. Tempesta, J Stat Mech P04021 (2010)

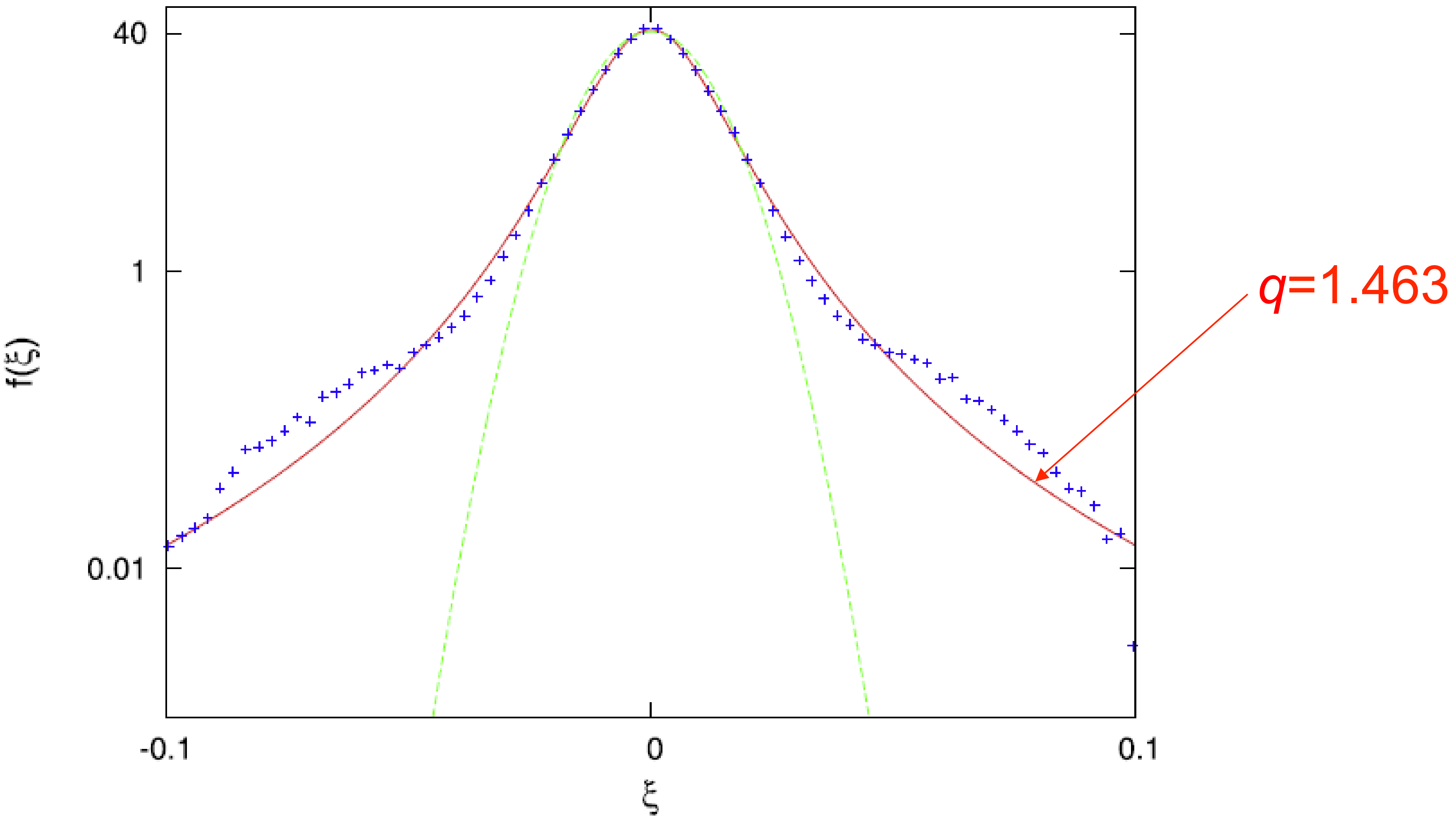

Figure 4. Plot on a linear-log scale of the numerical distribution $f(\xi)$ (blue points) fitted with a Tsallis distribution (red) and a Gauss distribution (green) for $N = 128$ and $\epsilon = 0.006$.

M. Leo, R.A. Leo and P. Tempesta, J Stat Mech P04021 (2010)

PHYSICAL REVIEW E **85**, 031149 (2012)

Non-Maxwellian behavior and quasistationary regimes near the modal solutions of the Fermi-Pasta-Ulam β system

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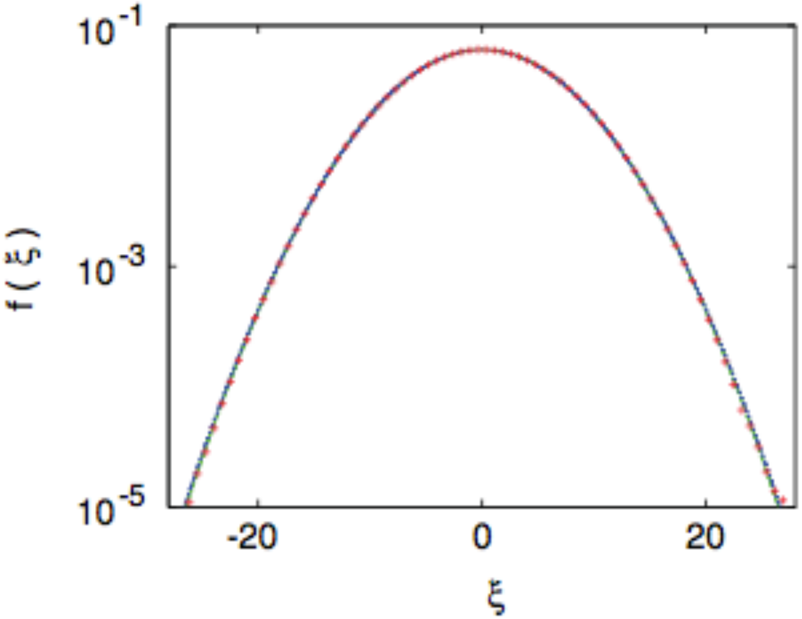
(Received 9 September 2011; published 30 March 2012)

In a recent paper [M. Leo, R. A. Leo, and P. Tempesta, J. Stat. Mech. (2011) P03003], it has been shown that the $\pi/2$-mode exact nonlinear solution of the Fermi-Pasta-Ulam β system, with periodic boundary conditions, admits two energy density thresholds. For values of the energy density ϵ below or above these thresholds, the solution is stable. Between them, the behavior of the solution is unstable, first recurrent and then chaotic. In this paper, we study the chaotic behavior between the two thresholds from a statistical point of view, by analyzing the distribution function of a dynamical variable that is zero when the solution is stable and fluctuates around zero when it is unstable. For mesoscopic systems clear numerical evidence emerges that near the second threshold, in a large range of the energy density, the numerical distribution is fitted accurately with a q-Gaussian distribution for very large integration times, suggesting the existence of a quasistationary state possessing a weakly chaotic behavior. A normal distribution is recovered in the thermodynamic limit.

$$H = \frac{1}{2}\sum_{i=1}^{N} {p_i}^2 + \frac{1}{2}\sum_{i=1}^{N}(x_{i+1}-x_i)^2 + \frac{\beta}{4}\sum_{i=1}^{N}(x_{i+1}-x_i)^4$$

with $x_{N+1} = x_1$ and $\beta \geqslant 0$. All quantities are dimensionless.



M. Leo, R.A. Leo, P. Tempesta and C. T.
Phys Rev E **85**, 031149 (2012)

FIG. 7. (Color online) log-linear representation of the three distributions for $N = 256$ and $\epsilon = 0.1477$: Gaussian and q-Gaussian distributions overlap with the numerical one (red crosses). We get $\chi_g^2 = 2.42 \times 10^{-7}$ and $\chi_{qg}^2 = 2.21 \times 10^{-7}$.

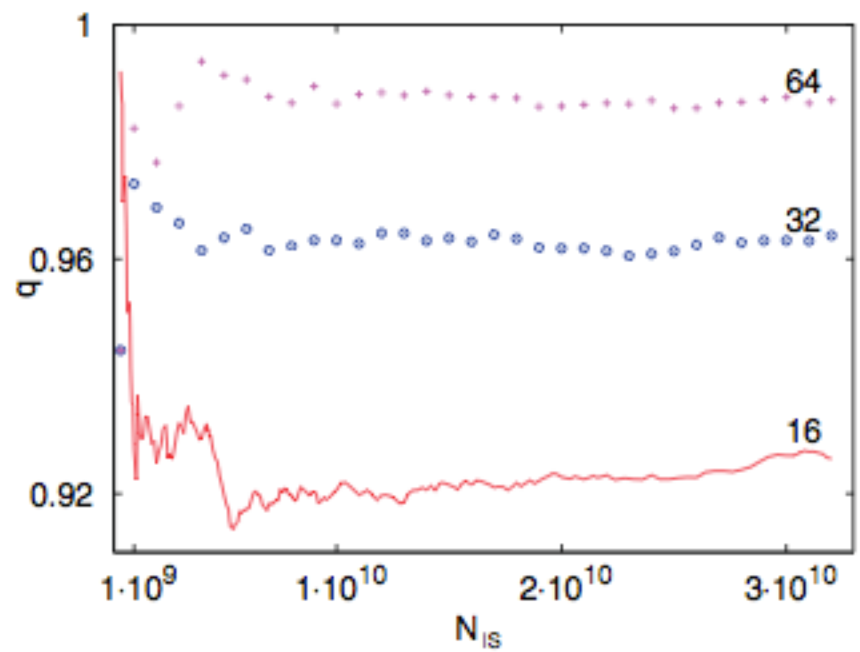
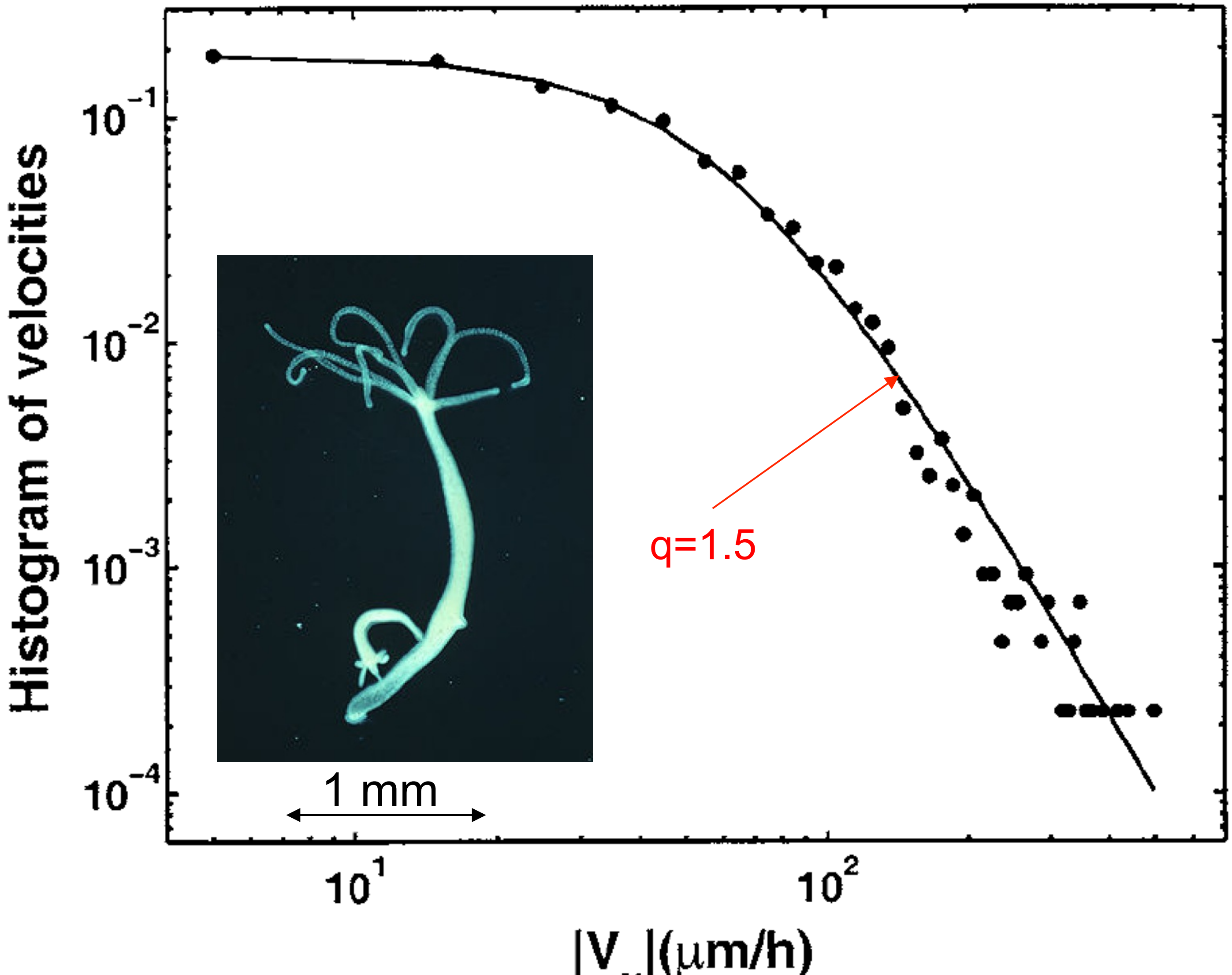
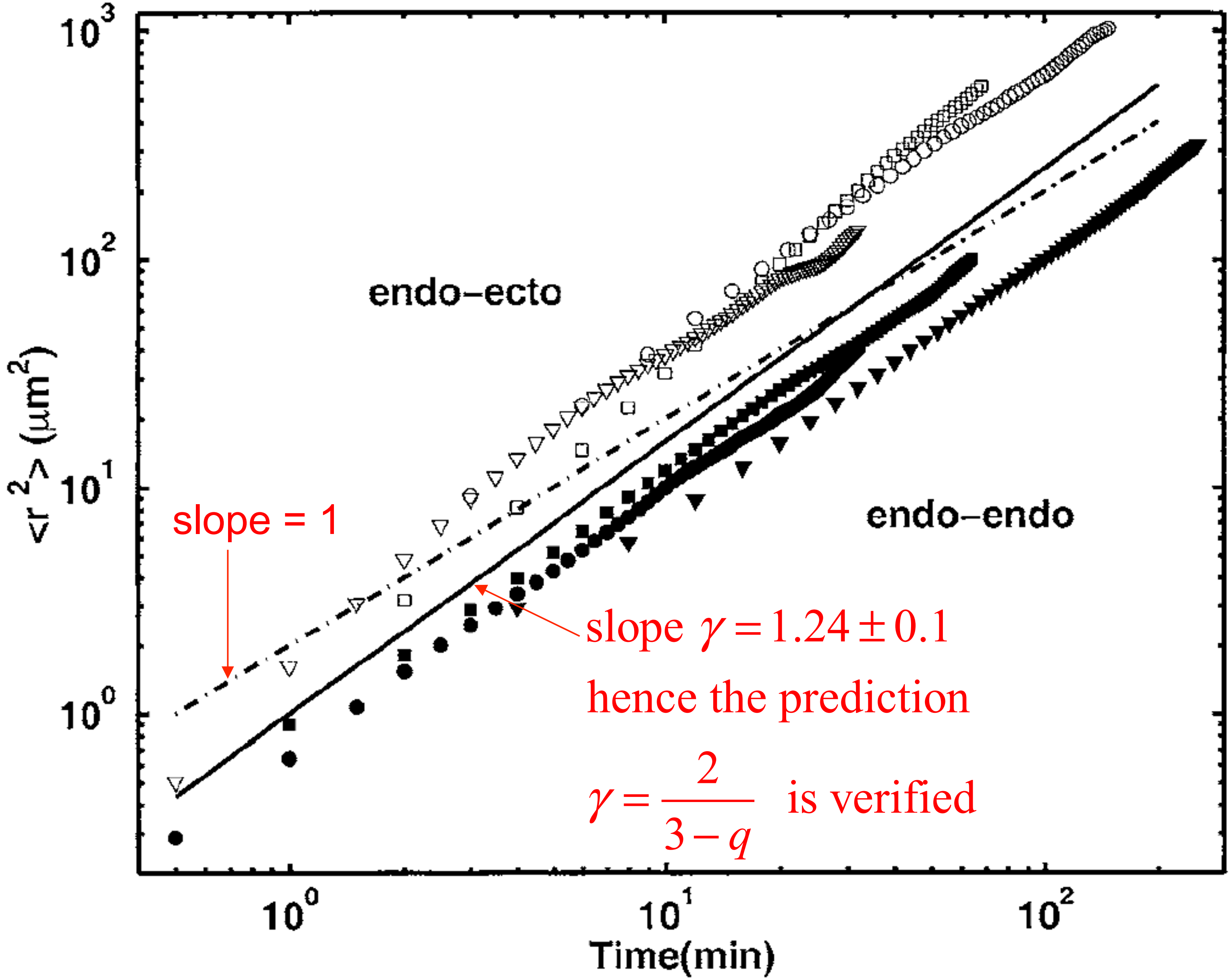


FIG. 9. (Color online) q vs the number of integration steps N_{I} $\epsilon = 0.095$, $N = 16$ (red solid curve), $N = 32$ (blue ○), and $N =$ (purple pluses). The integration time is $t = 0.02N_{IS}$. The asymp values of q are respectively $q_{16} = 0.9255 \pm 0.0003$, $q_{32} = 0.964$ 0.0004, $q_{64} = 0.9871 \pm 0.0004$.

Hydra viridissima:

A Upadhyaya, J-P Rieu, JA Glazier and Y Sawada, Physica A **293**, 549 (2001)





A Upadhyaya, J-P Rieu, JA Glazier and Y Sawada, Physica A **293**, 549 (2001)

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

RAPID COMM

PHYSICAL REVIEW A **67**, 051402(R) (2003)

Anomalous diffusion and Tsallis statistics in an optical lattice

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(Received 26 February 2003; published 27 May 2003)

We point out a connection between anomalous transport in an optical lattice and Tsallis' generalized statistics. Specifically, we show that the momentum equation for the semiclassical Wigner function which describes atomic motion in the optical potential, belongs to a class of transport equations recently studied by Borland [Phys. Lett. A **245**, 67 (1998)]. The important property of these ordinary linear Fokker-Planck equations is that their stationary solutions are exactly given by Tsallis distributions. An analytical expression of the Tsallis index q in terms of the microscopic parameters of the quantum-optical problem is given and the spatial coherence of the atomic wave packets is discussed.

(i) The distribution of atomic velocities is a *q*-Gaussian;

(ii) $q = 1 + \dfrac{44 E_R}{U_0}$ where $E_R \equiv$ recoil energy

$U_0 \equiv$ potential depth

Tunable Tsallis Distributions in Dissipative Optical Lattices

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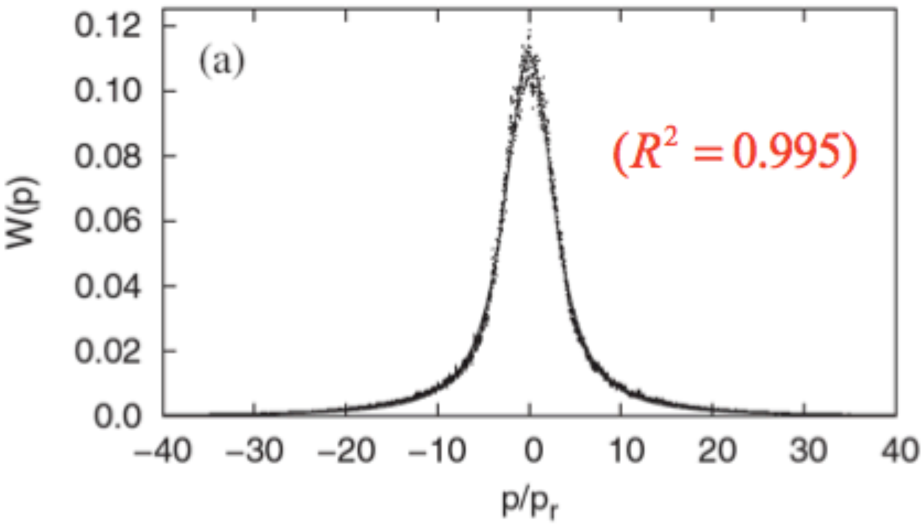
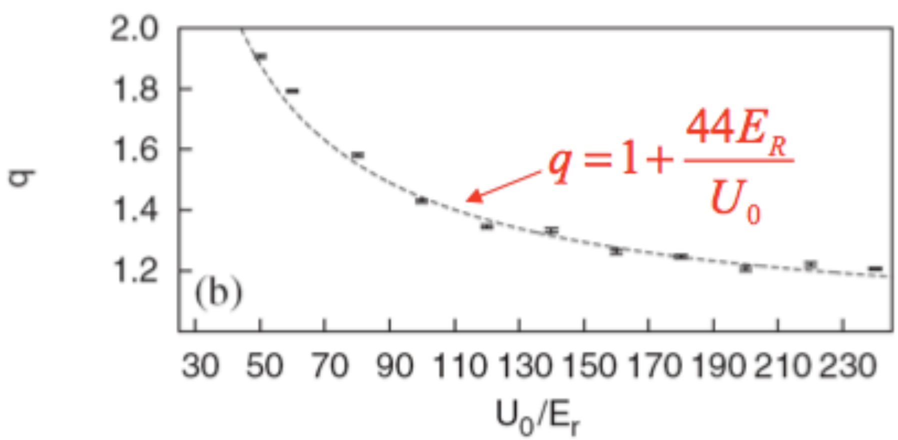
(Received 10 January 2006; published 24 March 2006)

We demonstrated experimentally that the momentum distribution of cold atoms in dissipative optical lattices is a Tsallis distribution. The parameters of the distribution can be continuously varied by changing the parameters of the optical potential. In particular, by changing the depth of the optical lattice, it is possible to change the momentum distribution from Gaussian, at deep potentials, to a power-law tail distribution at shallow optical potentials.

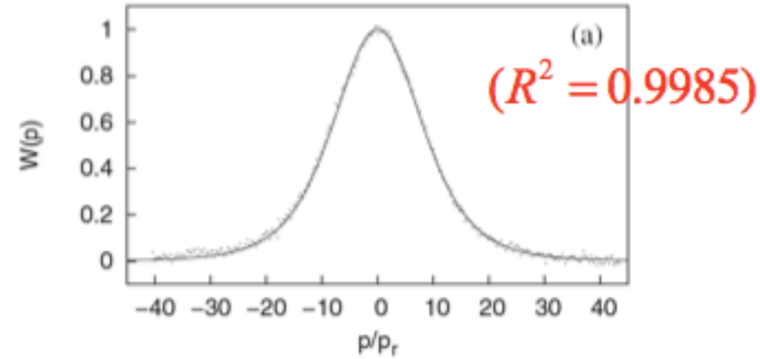
Experimental and computational verifications

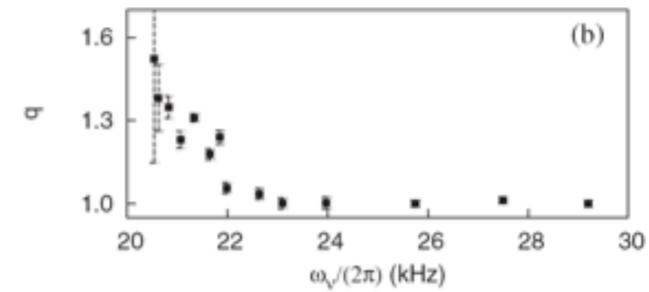
by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)

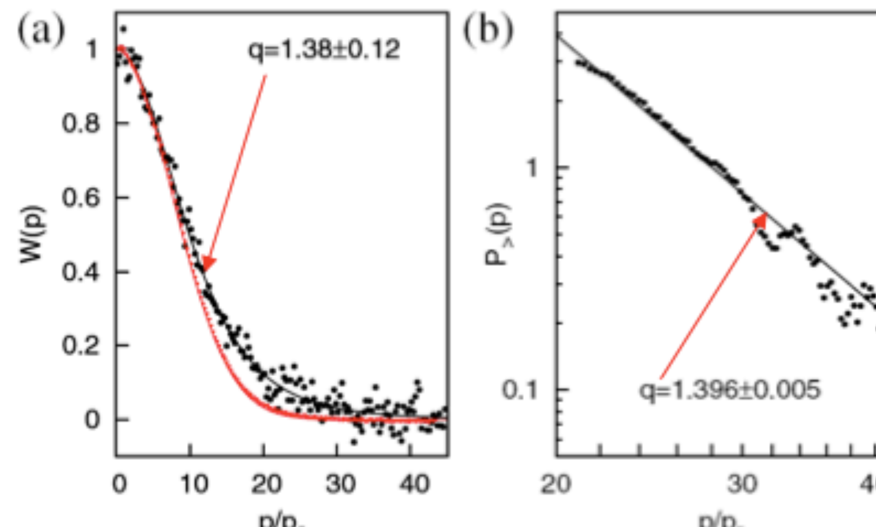




(Computational verification:
quantum Monte Carlo simulations)







(Experimental verification: Cs atoms)

PRL **100,** 055003 (2008) PHYSICAL REVIEW LETTERS week ending 8 FEBRUARY 2008

Superdiffusion and Non-Gaussian Statistics in a Driven-Dissipative 2D Dusty Plasma

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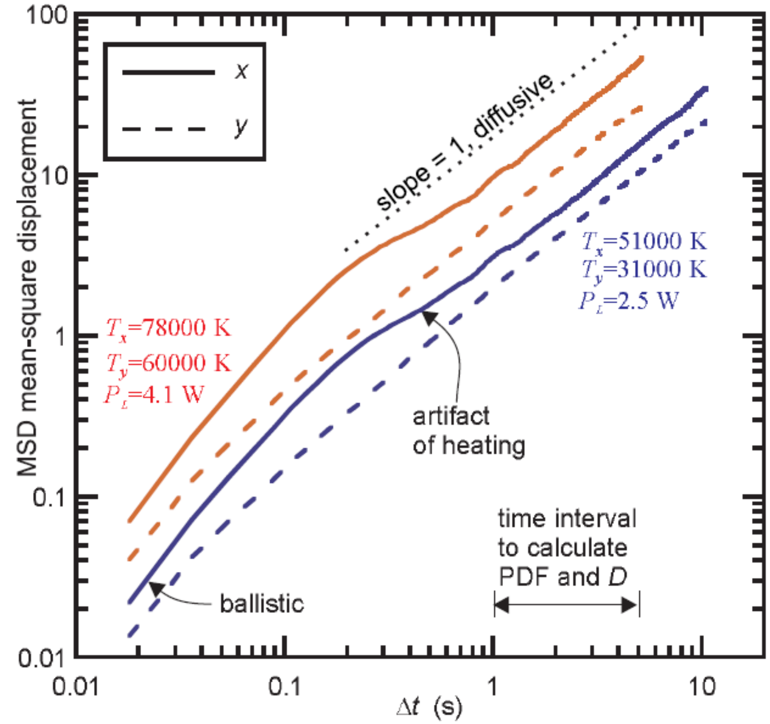
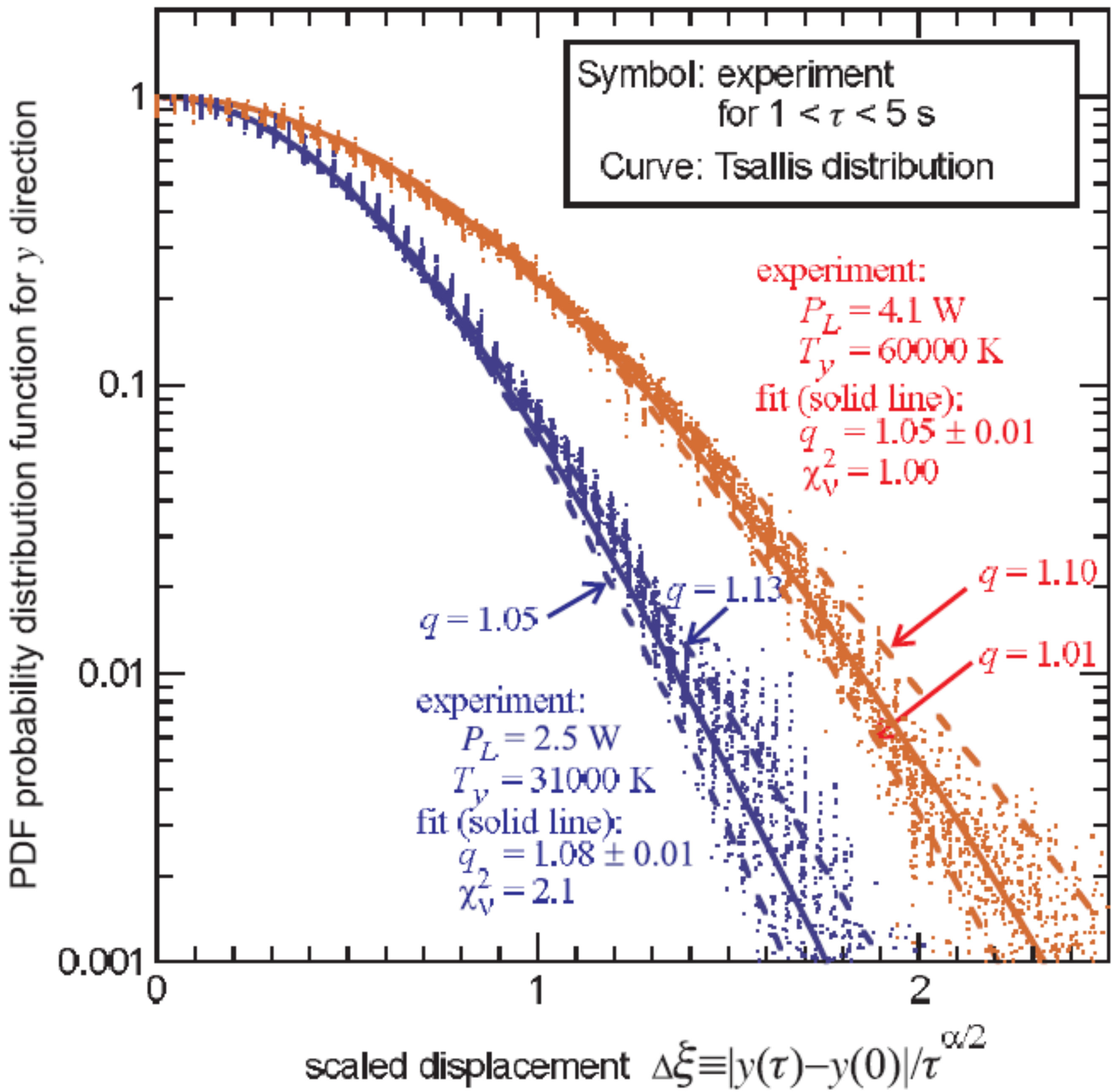
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA

(Received 1 June 2007; published 6 February 2008)

Anomalous diffusion and non-Gaussian statistics are detected experimentally in a two-dimensional driven-dissipative system. A single-layer dusty plasma suspension with a Yukawa interaction and frictional dissipation is heated with laser radiation pressure to yield a structure with liquid ordering. Analyzing the time series for mean-square displacement, superdiffusion is detected at a low but statistically significant level over a wide range of temperatures. The probability distribution function fits a Tsallis distribution, yielding q, a measure of nonextensivity for non-Gaussian statistics.

$$< r^2 > \;\propto\; t^{\alpha}$$





B. Liu and J. Goree, Phys Rev Lett **100**, 055003 (2008)

LASER COOLING:

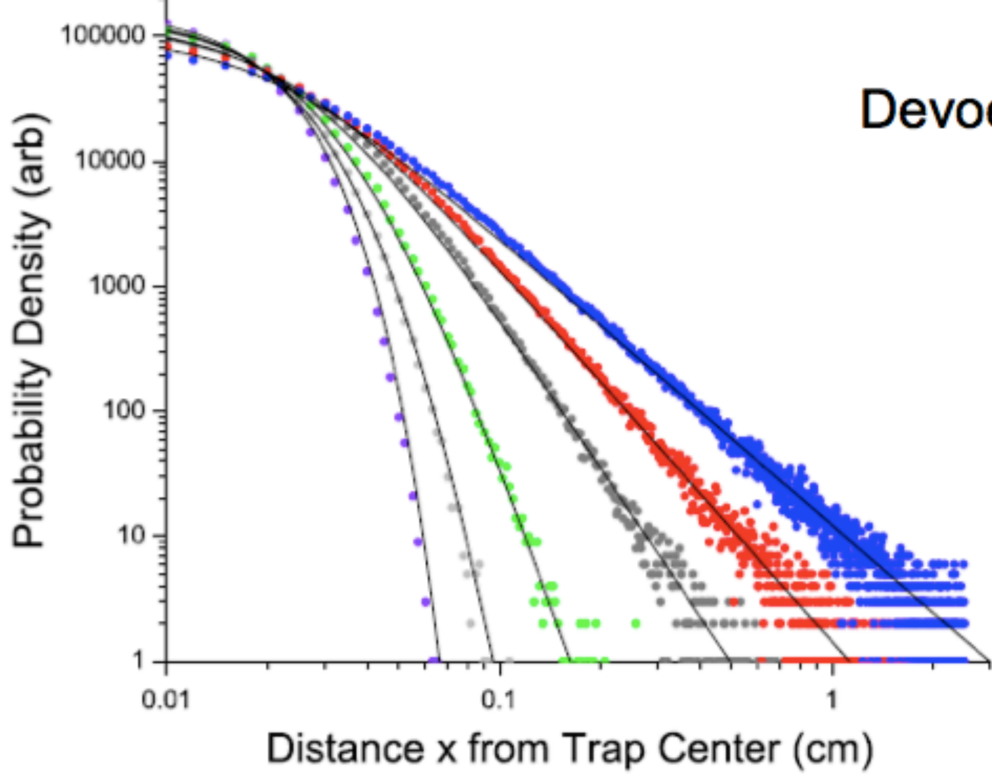
PRL **102,** 063001 (2009) PHYSICAL REVIEW LETTERS week ending 13 FEBRUARY 2009

Power-Law Distributions for a Trapped Ion Interacting with a Classical Buffer Gas

Ralph G. DeVoe
Physics Department, Stanford University, Stanford, California 94305, USA
(Received 3 November 2008; published 10 February 2009)

Classical collisions with an ideal gas generate non-Maxwellian distribution functions for a single ion in a radio frequency ion trap. The distributions have power-law tails whose exponent depends on the ratio of buffer gas to ion mass. This provides a statistical explanation for the previously observed transition from cooling to heating. Monte Carlo results approximate a Tsallis distribution over a wide range of parameters and have *ab initio* agreement with experiment.

Devoe, Phys Rev Lett **102** (2009) 063001



$$T(x) = \frac{T(0)}{\left[1+(q-1)\left(\frac{x}{\sigma}\right)^2\right]^{\frac{1}{q-1}}}$$

FIG. 1 (color online). Monte Carlo distributions for a single $^{136}Ba^+$ ion cooled by six different buffer gases at 300 K ranging from $m_B = 4$ (left) to $m_B = 200$ (right). Note the evolution from Gaussian to power law (straight line) as the mass increases. The solid lines are Tsallis functions [Eq. (7)] with fixed $\sigma = 0.0185$ cm and the exponents of Table I.

TABLE I. Tsallis parameters n and q_T fit from Fig. 1.

Buffer gas	m_I/m_B	n	q_T
He	34.5	>60	1.03
Ar	3.40	8.2	1.12
Kr	1.70	3.8	1.26
Xe	1.0	1.98	1.51
170	0.80	1.50	1.80
200	0.68	1.15	1.87

Prediction of the q - triplet: C. T., Physica A **340**,1 (2004)

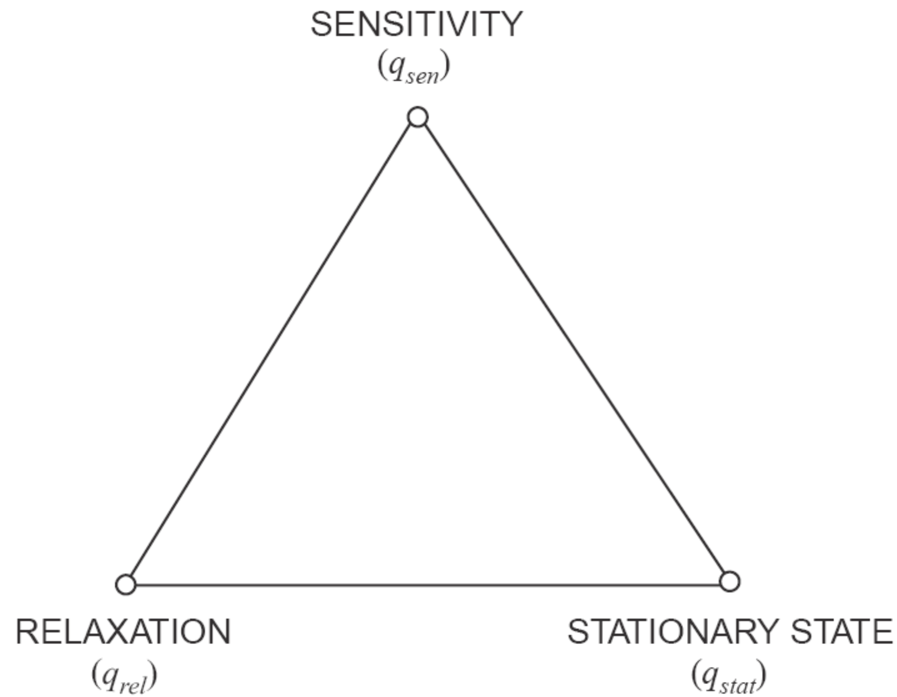


Fig. 2. The triangle of the basic values of q, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leqslant 1$, $q_{rel} \geqslant 1$ and $q_{stat} \geqslant 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent α and the dimension d, it could be that q_{stat} decreases from a value above unity (e.g., 2 or $\frac{3}{2}$) to unity when α/d increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for q_{sen} and q_{rel}, but only those associated with a Hamiltonian are expected to *also* have a well defined value for q_{stat}.

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Physica A 356 (2005) 375–384

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Triangle for the entropic index q of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere

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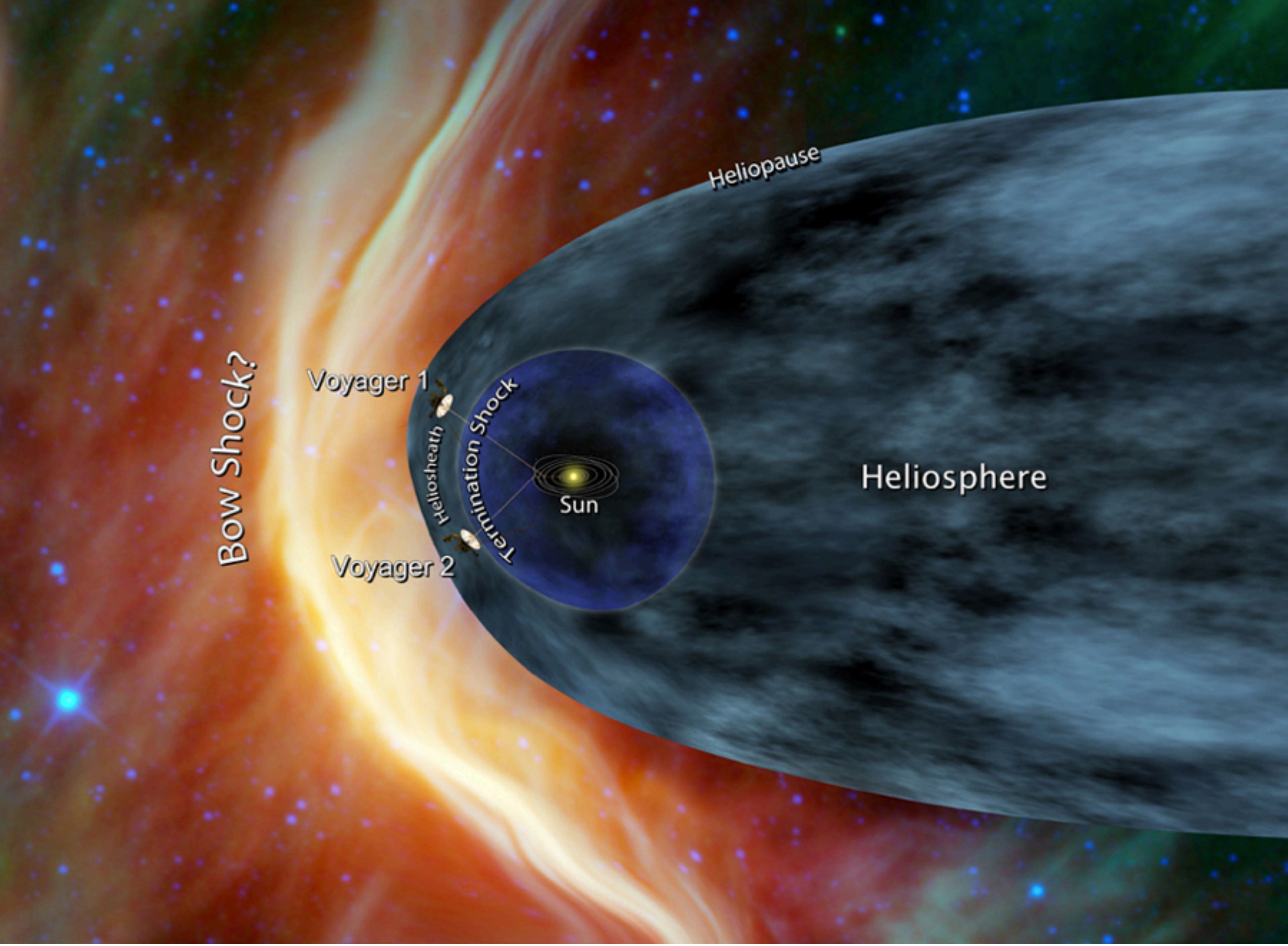
Laboratory for Solar and Space Physics, Code 612.2, NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA

Received 10 June 2005
Available online 11 July 2005

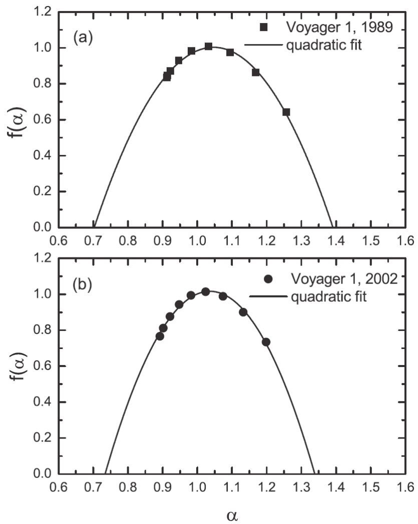


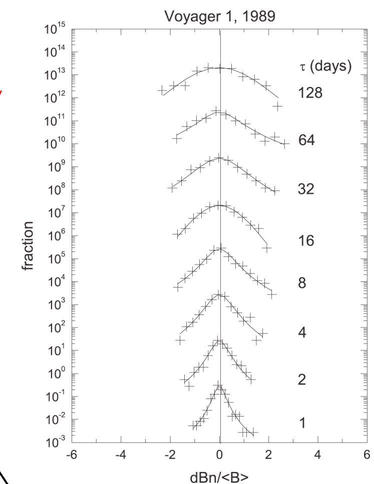
Heliopause
Bow Shock?
Voyager 1
Voyager 2
Heliosheath
Termination Shock
Sun
Heliosphere

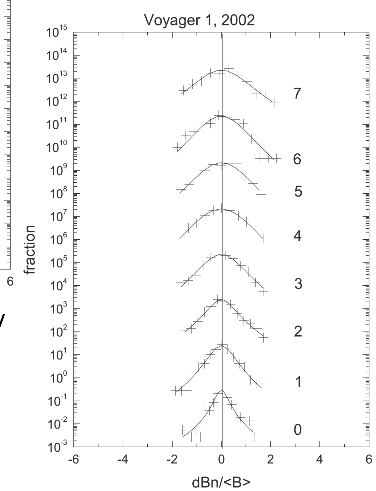
SOLAR WIND: Magnetic Field Strength

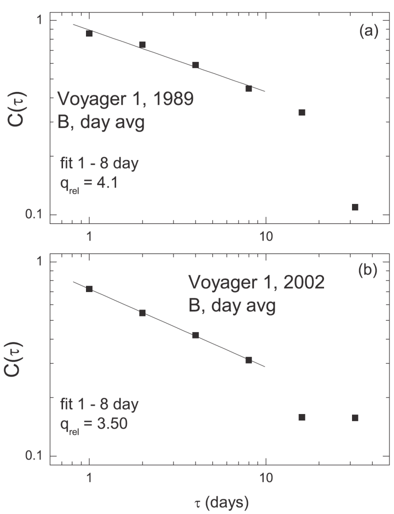
L.F. Burlaga and A. F.-Vinas (2005) / NASA Goddard Space Flight Center; Physica A **356**, 375 (2005)

[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]









$$q_{sen} = -0.6 \pm 0.2$$

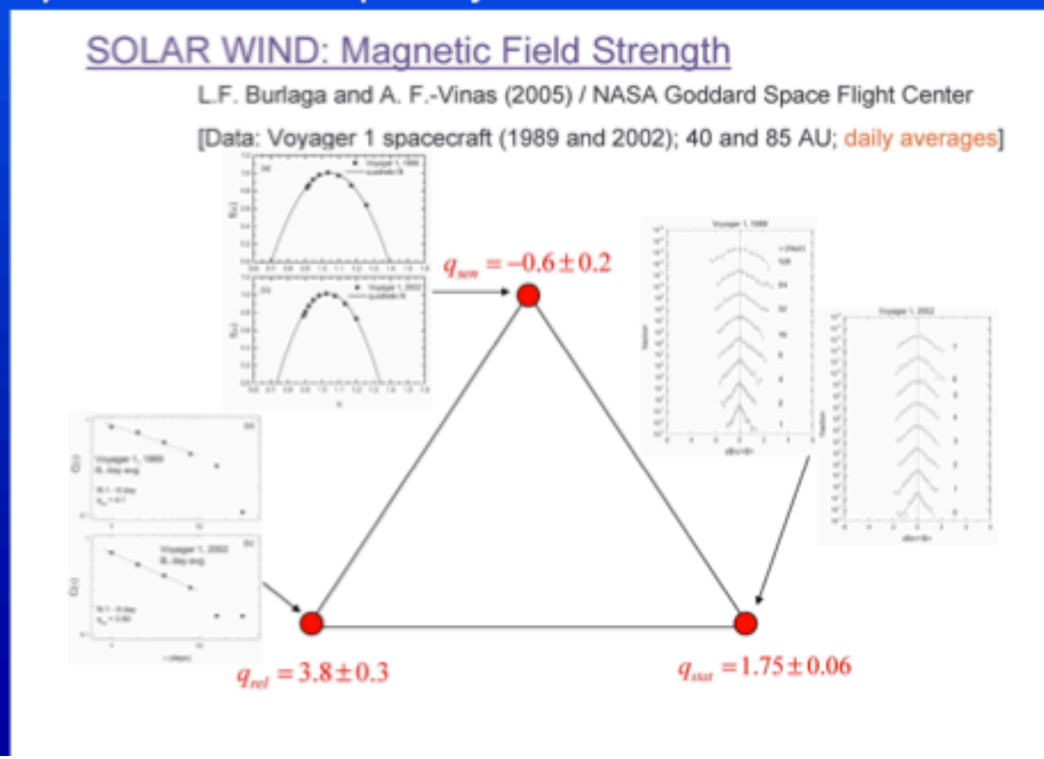
$$q_{rel} = 3.8 \pm 0.3$$

$$q_{stat} = 1.75 \pm 0.06$$

$$q_{sens} < 1 < q_{stat} < q_{rel}$$

IHY 2007: VOYAGER 1: Fundamental Physics

The atmosphere of the Sun beyond a few solar radii, known as HELIOSPHERE, is fully ionized plasma expanding at supersonic speeds, carrying solar magnetic fields with it. This solar wind is a driven non-linear non-equilibrium system. The Sun injects matter, momentum, energy, and magnetic fields into the heliosphere in a highly variable way. Voyager 1 observed magnetic field strength variations in the solar wind near 40 AU during 1989 and near 85 AU during 2002. Tsallis' non-extensive statistical mechanics, a generalization of Boltzmann-Gibbs statistical mechanics, allows a physical explanation of these magnetic field strength variations in terms of departure from thermodynamic equilibrium in an unique way:



Computers and Mathematics with Applications 60 (2010) 2426–2432

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A novel automatic microcalcification detection technique using Tsallis entropy & a type II fuzzy index

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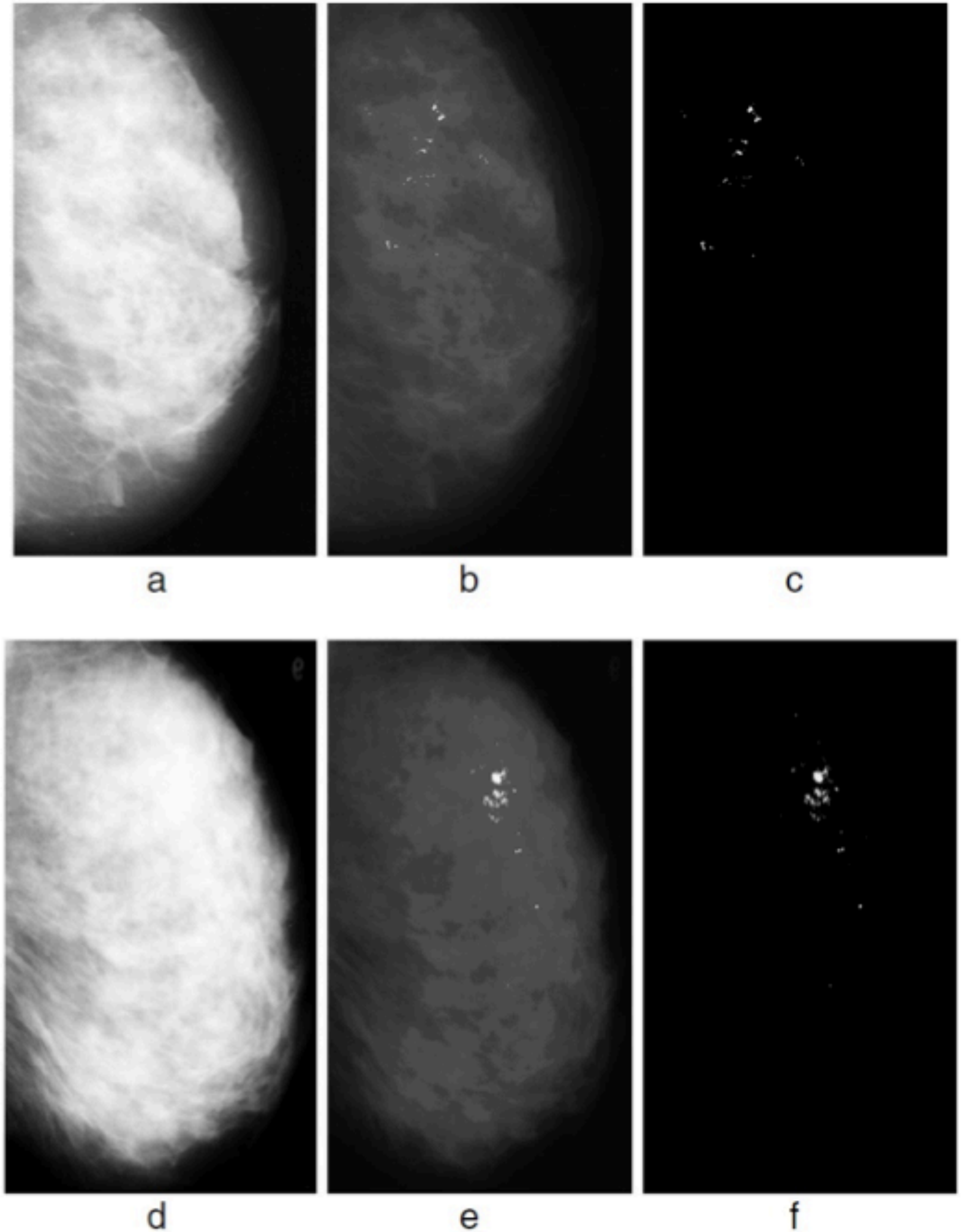
Type II fuzzy index
Shannon entropy
Mammograms
Microcalcification

ABSTRACT

This article investigates a novel automatic microcalcification detection method using a type II fuzzy index. The thresholding is performed using the Tsallis entropy characterized by another parameter '*q*', which depends on the non-extensiveness of a mammogram. In previous studies, '*q*' was calculated using the histogram distribution, which can lead to erroneous results when pectoral muscles are included. In this study, we have used a type II fuzzy index to find the optimal value of '*q*'. The proposed approach has been tested on several mammograms. The results suggest that the proposed Tsallis entropy approach outperforms the two-dimensional non-fuzzy approach and the conventional Shannon entropy partition approach. Moreover, our thresholding technique is completely automatic, unlike the methods of previous related works. Without Tsallis entropy enhancement, detection of microcalcifications is meager: 80.21% Tps (true positives) with 8.1 Fps (false positives), whereas upon introduction of the Tsallis entropy, the results surge to 96.55% Tps with 0.4 Fps.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Weili SHI, Yanfang LI, Yu MIAO, Yinlong HU

Changchun University of Science and Technology

Research on the Key Technology of Image Guided Surgery

***Abstract**. It research on the key technology on IGS (image-guided surgery). It proposes medical image segmentation based on PCNN and the virtual endoscopic scenes real-time rendering method based on GPU parallel computing technology, which improves the display quality of IGS's virtual scene and real-time rendering speed. These methods are very important for IGS's applications.*

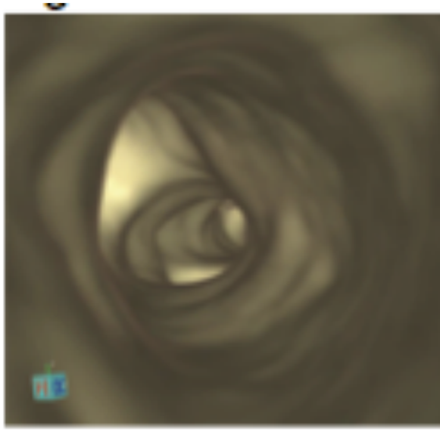
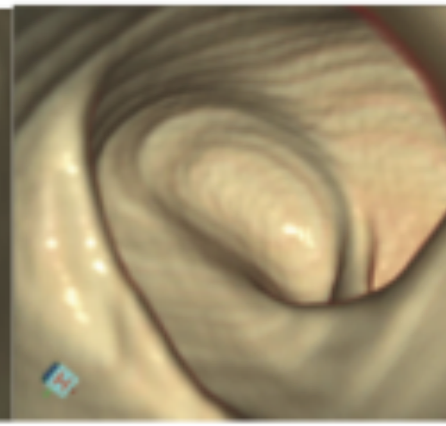
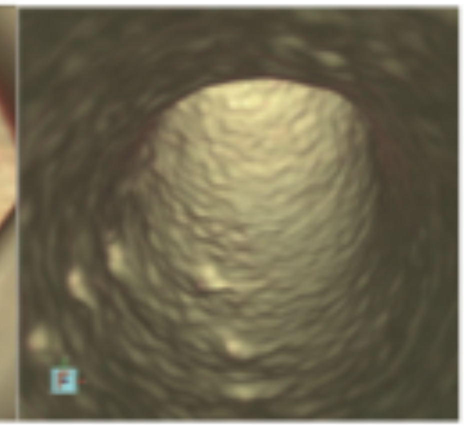

Fig.12. Bronchus Fig.13.Colon Fig.14.blood vessel

Table 1. Speed Comparison between Traditional Algorithms and Present Algorithm(uint: fps)

CT Image	Bronchus	Colon	blood vessel
Image Extent	512*51*217	512*512*252	512*512*355
Ray casting	9.8	6.4	3.5
Our algorithm	36.2	35.7	32.1

Tissue Radiation Response with Maximum Tsallis Entropy

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J. C. Antoranz

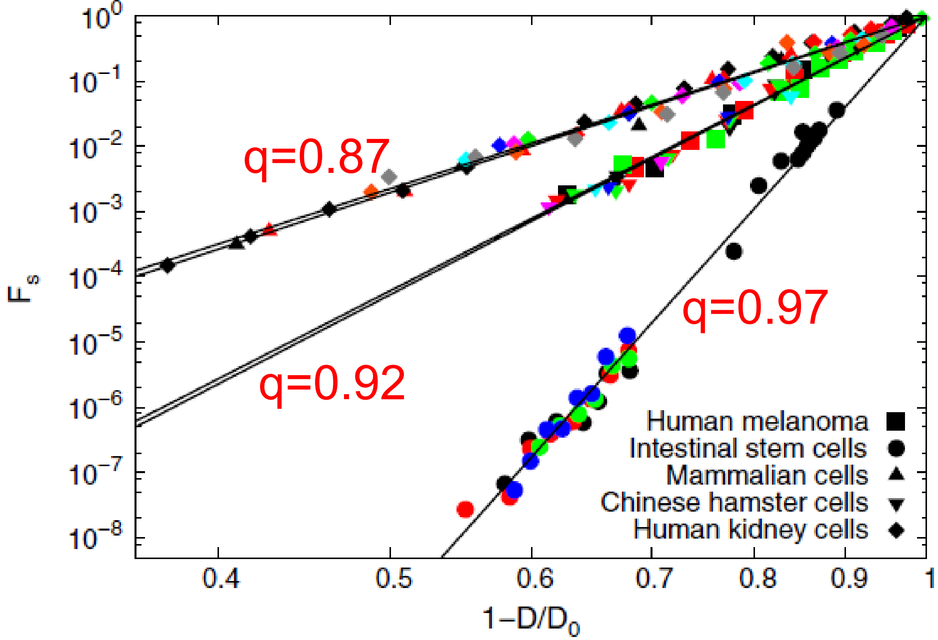
UNED, Departamento de Física Matemática y de Fluidos, 28040 Madrid, Spain, and University of Havana, Cátedra de Sistemas Complejos Henri Poincaré, Havana 10400, Cuba

Oscar Sotolongo-Costa

University of Havana, Cátedra de Sistemas Complejos Henri Poincaré, Havana 10400, Cuba

(Received 22 June 2010; published 7 October 2010)

The expression of survival factors for radiation damaged cells is currently based on probabilistic assumptions and experimentally fitted for each tumor, radiation, and conditions. Here, we show how the simplest of these radiobiological models can be derived from the maximum entropy principle of the classical Boltzmann-Gibbs expression. We extend this derivation using the Tsallis entropy and a cutoff hypothesis, motivated by clinical observations. The obtained expression shows a remarkable agreement with the experimental data found in the literature.



$$\gamma = \frac{2-q}{1-q}$$

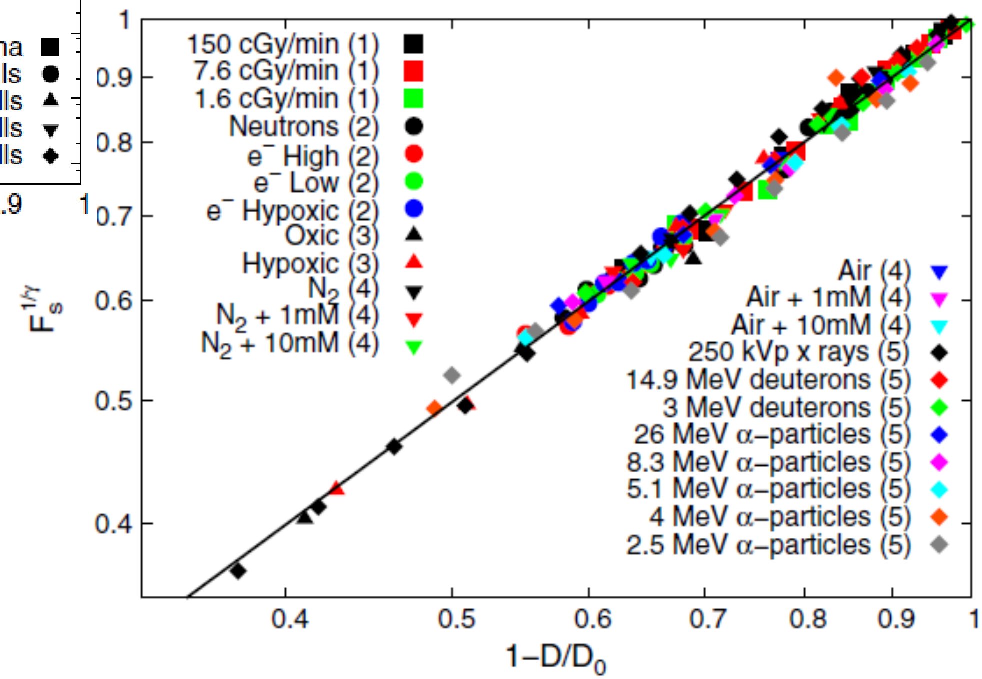


FIG. 2 (color online). Normalized survival fractions $(F_s)^{1/\gamma}$ as a function of the rescaled radiation dose, $1 - D/D_0$ for different tissues: intestinal stem cells (■), chinese hamster cells (●), human melanoma (▲), human kidney cells (▼), and cultured mammalian cells (◆) under different irradiation conditions detailed in [17–21] and grouped in [23]. The straight line shown is $y = x$.

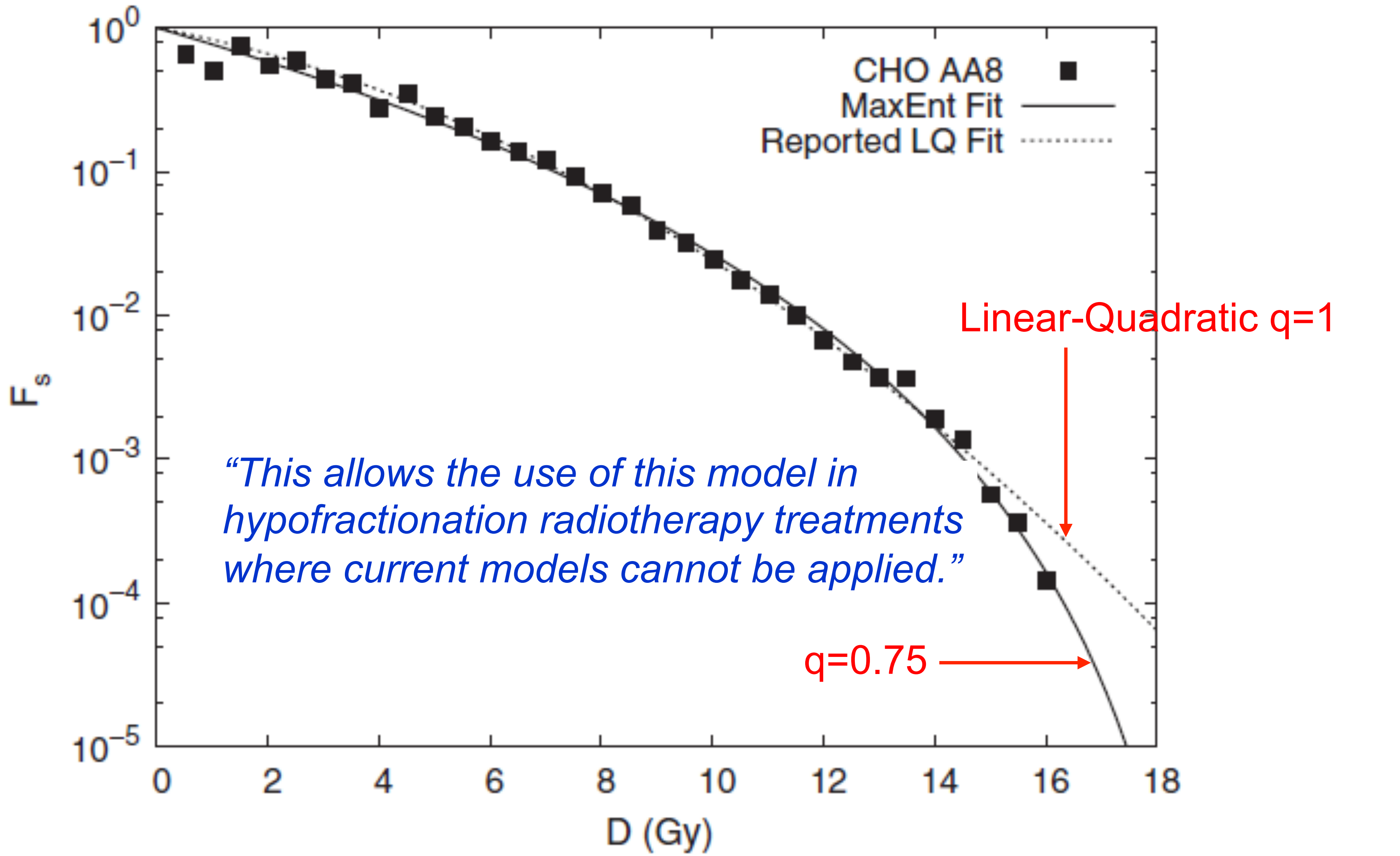


FIG. 3. Comparison between the LQ model best fit ($\alpha = 0.167 \pm 0.015\ \mathrm{Gy}^{-1}$ and $\beta = 0.0205 \pm 0.0015\ \mathrm{Gy}^{-2}$) reported in [24] and our model fitted to $\gamma = 5.0 \pm 0.4$ and $D_0 = 19.4 \pm 0.4$ Gy for the cell line CHO AA8 under 250 k-Vp x rays.

Limoges - France

Strain-profile determination in ion-implanted single crystals using generalized simulated annealing

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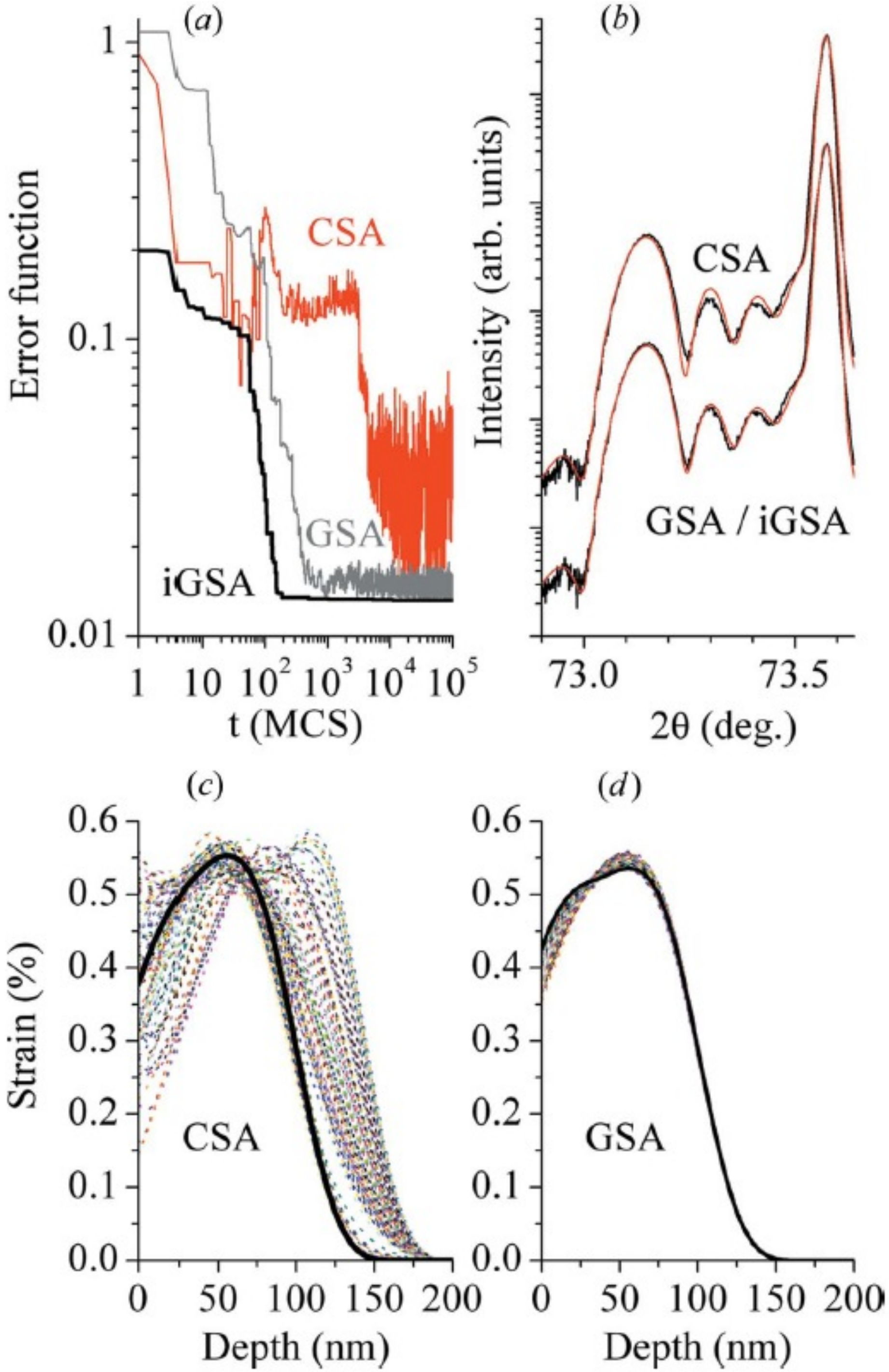
A novel least-squares fitting procedure is presented that allows the retrieval of strain profiles in ion-implanted single crystals using high-resolution X-ray diffraction. The model is based on the dynamical theory of diffraction, including a B-spline-based description of the lattice strain. The fitting procedure relies on the generalized simulated annealing algorithm which, contrarily to most common least-squares fitting-based methods, allows the global minimum of the error function (the difference between the experimental and the calculated curves) to be found extremely quickly. It is shown that convergence can be achieved in a few hundred Monte Carlo steps, *i.e.* a few seconds. The method is model-independent and allows determination of the strain profile even without any 'guess' regarding its shape. This procedure is applied to the determination of strain profiles in Cs-implanted yttria-stabilized zirconia (YSZ). The strain and damage profiles of YSZ single crystals implanted at different ion fluences are analyzed and discussed.



(a)
(b)
Error function
1
0.1
0.01
CSA
GSA
iGSA
1
10
10^2
10^3
10^4
10^5
t (MCS)
Intensity (arb. units)
CSA
GSA / iGSA
73.0
73.5
2θ (deg.)
(c)
(d)
Strain (%)
0.6
0.5
0.4
0.3
0.2
0.1
0.0
CSA
GSA
0
50
100
150
200
Depth (nm)

Physics Reports 515 (2012) 115–226

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Physical approach to complex systems

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Typically, complex systems are natural or social systems which consist of a large number of nonlinearly interacting elements. These systems are open, they interchange information or mass with environment and constantly modify their internal structure and patterns of activity in the process of self-organization. As a result, they are flexible and easily adapt to variable external conditions. However, the most striking property of such systems is the existence of emergent phenomena which cannot be simply derived or predicted solely from the knowledge of the systems' structure and the interactions among their individual elements. This property points to the holistic approaches which require giving parallel descriptions of the same system on different levels of its organization. There is strong evidence – consolidated also in the present review – that different, even apparently disparate complex systems can have astonishingly similar characteristics both in their structure and in their behaviour. One can thus expect the existence of some common, universal laws that govern their properties.

Physics methodology proves helpful in addressing many of the related issues. In this review, we advocate some of the computational methods which in our opinion are especially fruitful in extracting information on selected – but at the same time most representative – complex systems like human brain, financial markets and natural language, from the time series representing the observables associated with these systems. The properties we focus on comprise the collective effects and their coexistence with noise, long-range interactions, the interplay between determinism and flexibility in evolution, scale invariance, criticality, multifractality and hierarchical structure. The methods described either originate from "hard" physics – like the random matrix theory – and then were transmitted to other fields of science via the field of complex systems research, or they originated elsewhere but turned out to be very useful also in physics — like, for example, fractal geometry. Further methods discussed borrow from the formalism of complex networks, from the theory of critical phenomena and from nonextensive statistical mechanics. Each of these methods is helpful in analyses of specific aspects of complexity and all of them are mutually complementary.

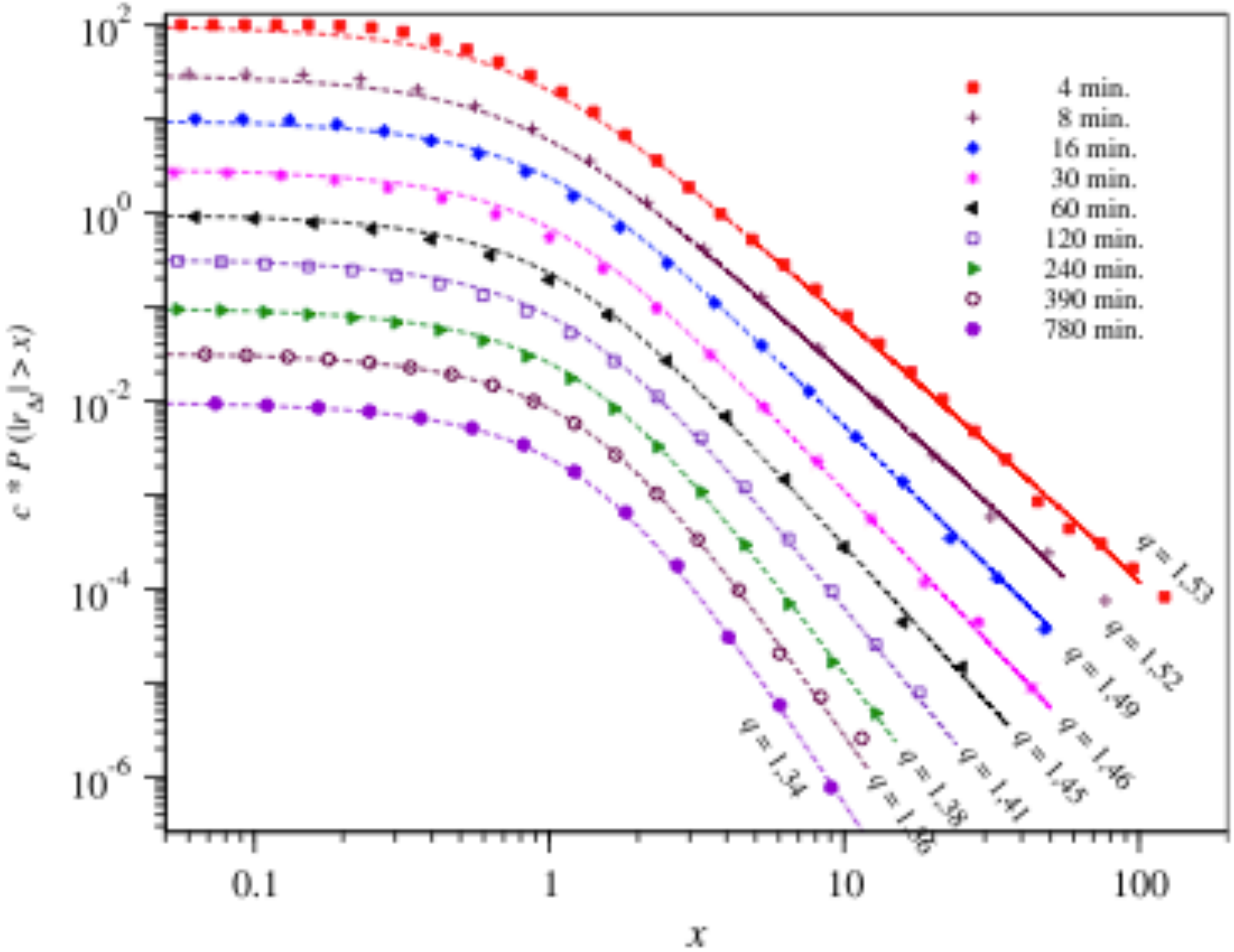
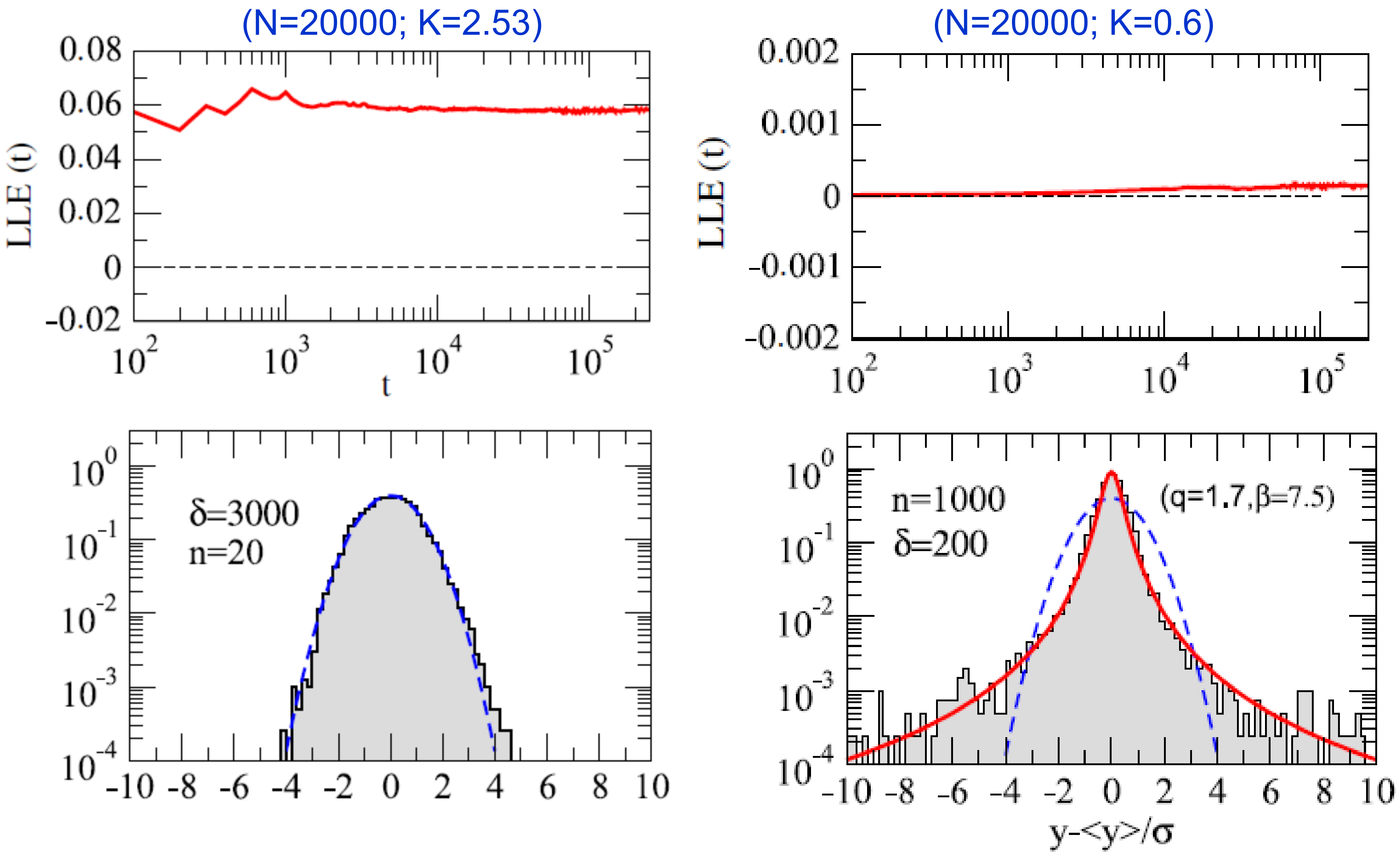


Fig. 61. Cumulative distributions of absolute normalized returns corresponding to different time scales Δt for the 100 American companies with the highest market capitalization, together with the fitted cumulative q-Gaussian distributions. Each q-Gaussian was labelled by the associated value of the Tsallis parameter q. In order to better visualize the results, each q-Gaussian was multiplied by a positive factor $c \neq 1$.

KURAMOTO MODEL: (N nonlinearly coupled oscillators)



G. Miritello, A. Pluchino and A. Rapisarda, Physica A **388**, 4818 (2009)

Thermostatistics of Overdamped Motion of Interacting Particles

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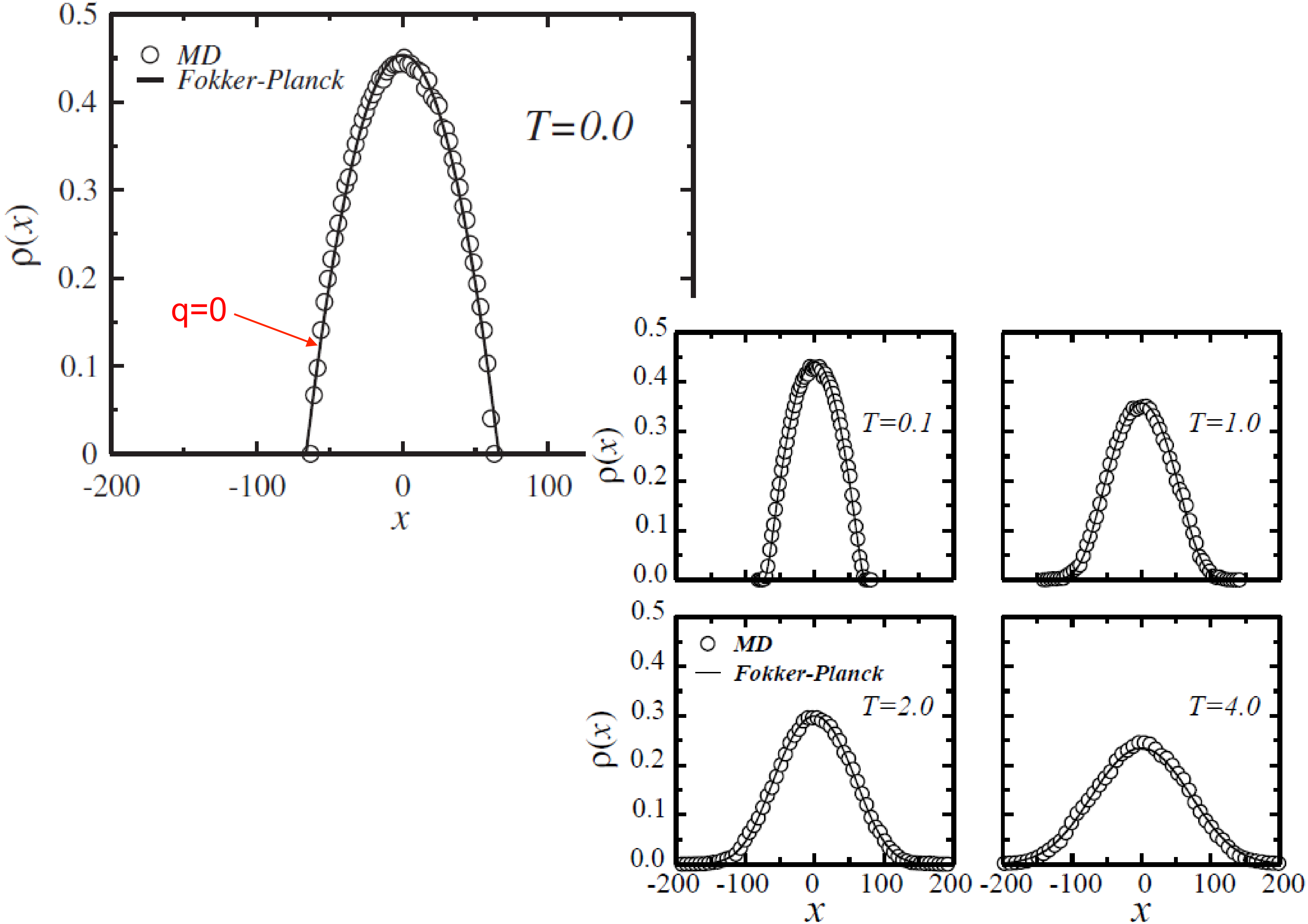
(Received 8 August 2010; published 22 December 2010)

We show through a nonlinear Fokker-Planck formalism, and confirm by molecular dynamics simulations, that the overdamped motion of interacting particles at $T = 0$, where T is the temperature of a thermal bath connected to the system, can be directly associated with Tsallis thermostatistics. For sufficiently high values of T, the distribution of particles becomes Gaussian, so that the classical Boltzmann-Gibbs behavior is recovered. For intermediate temperatures of the thermal bath, the system displays a mixed behavior that follows a novel type of thermostatistics, where the entropy is given by a linear combination of Tsallis and Boltzmann-Gibbs entropies.



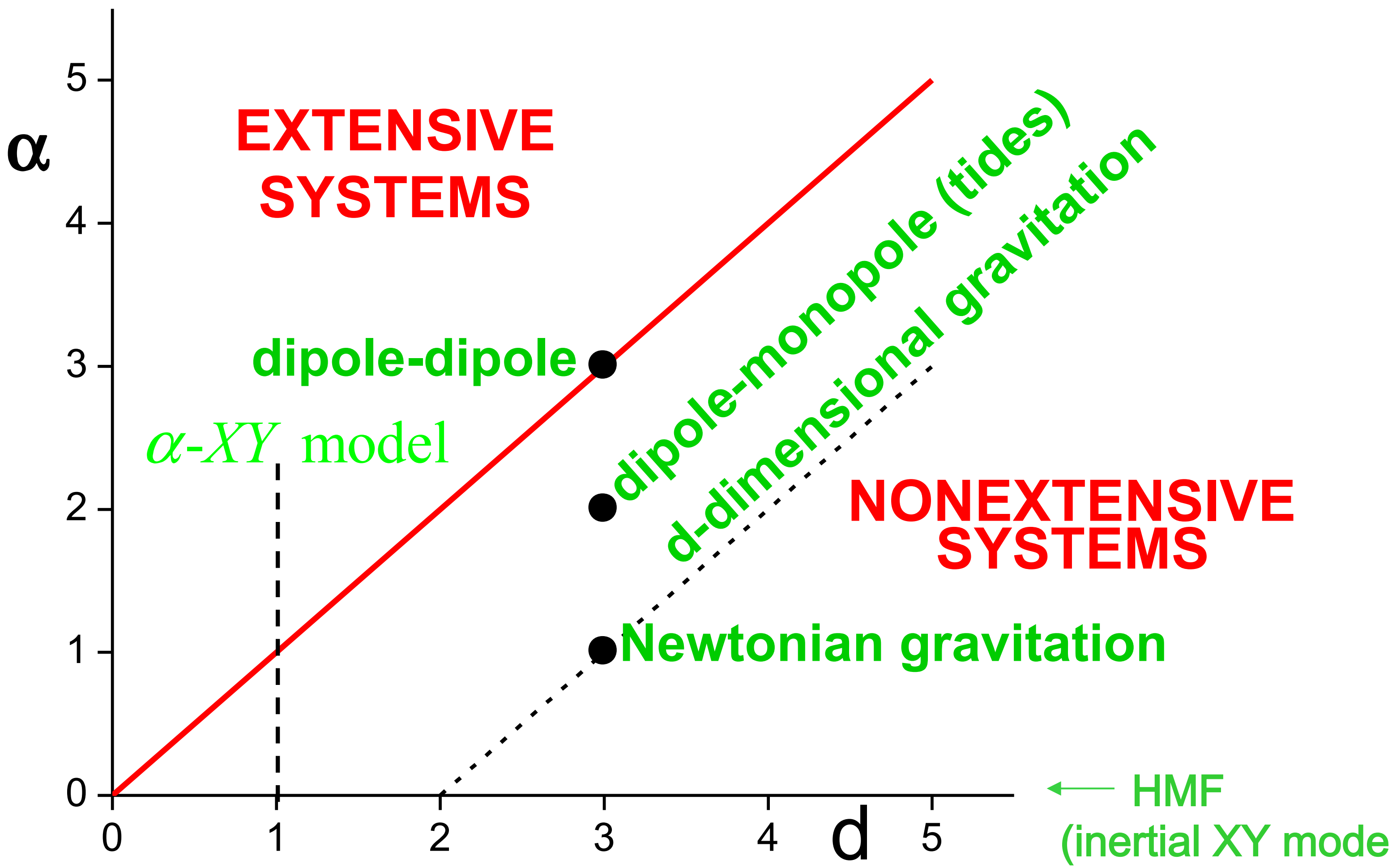
Andrade, Silva, Moreira, Nobre and Curado, Phys Rev Lett **105**, 260601 (2010)

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

$$V(r) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty) \qquad (A > 0,\ \ \alpha \geq 0)$$

integrable if $\alpha / d > 1$ *(short-ranged)*

non-integrable if $0 \leq \alpha / d \leq 1$ *(long-ranged)*



Influence of the interaction range on the thermostatistics of a classical many-body system

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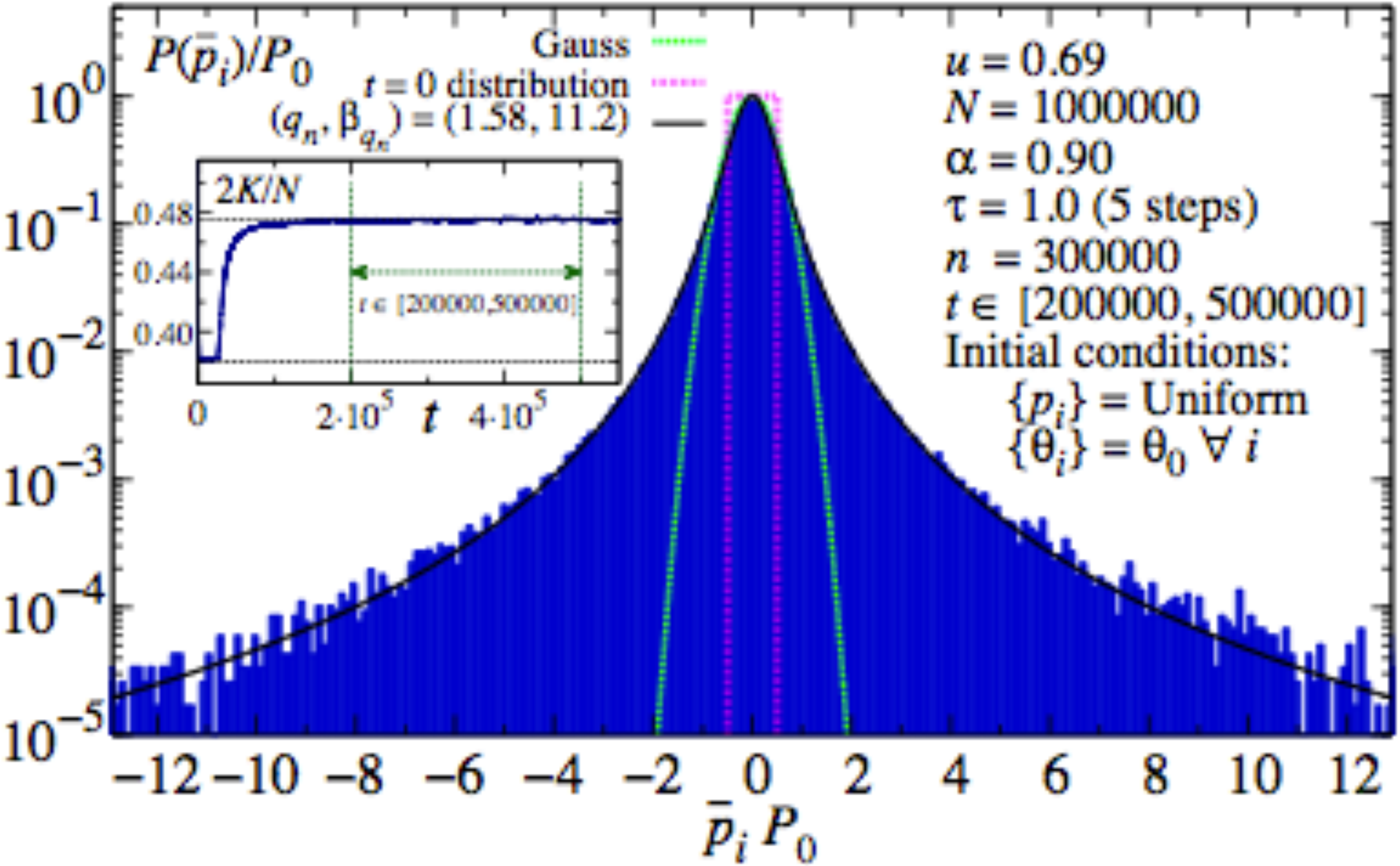
We numerically study a one-dimensional system of N classical localized planar rotators coupled through interactions which decay with distance as $1/r^\alpha$ ($\alpha \geq 0$). The approach is a first principle one (i.e., based on Newton's law) which, through molecular dynamics, yields the probability distribution of angular momenta. For α large enough we observe, for longstanding states corresponding to $N \gg 1$ systems, the expected Maxwellian distribution. But, for α small or comparable to unity, we observe instead robust fat-tailed distributions that are quite well fitted with q-Gaussians. These distributions extremize, under appropriate simple constraints, the nonadditive entropy S_q upon which nonextensive statistical mechanics is based. The whole scenario appears to be consistent with nonergodicity and with the q-generalized Central Limit Theorem. It confirms the more-than-centennial prediction by J.W. Gibbs that standard statistical mechanics are not applicable for long-range interactions (i.e., for $0 \leq \alpha \leq 1$) due to the divergence of the canonical partition function.

arXiv:1206.6133v1 [cond-mat.stat-mech] 26 Jun 2012



L.J.L. Cirto, V.R.V. Assis and C. T. (2012),1206.6133 [cond-mat.stat-mech]

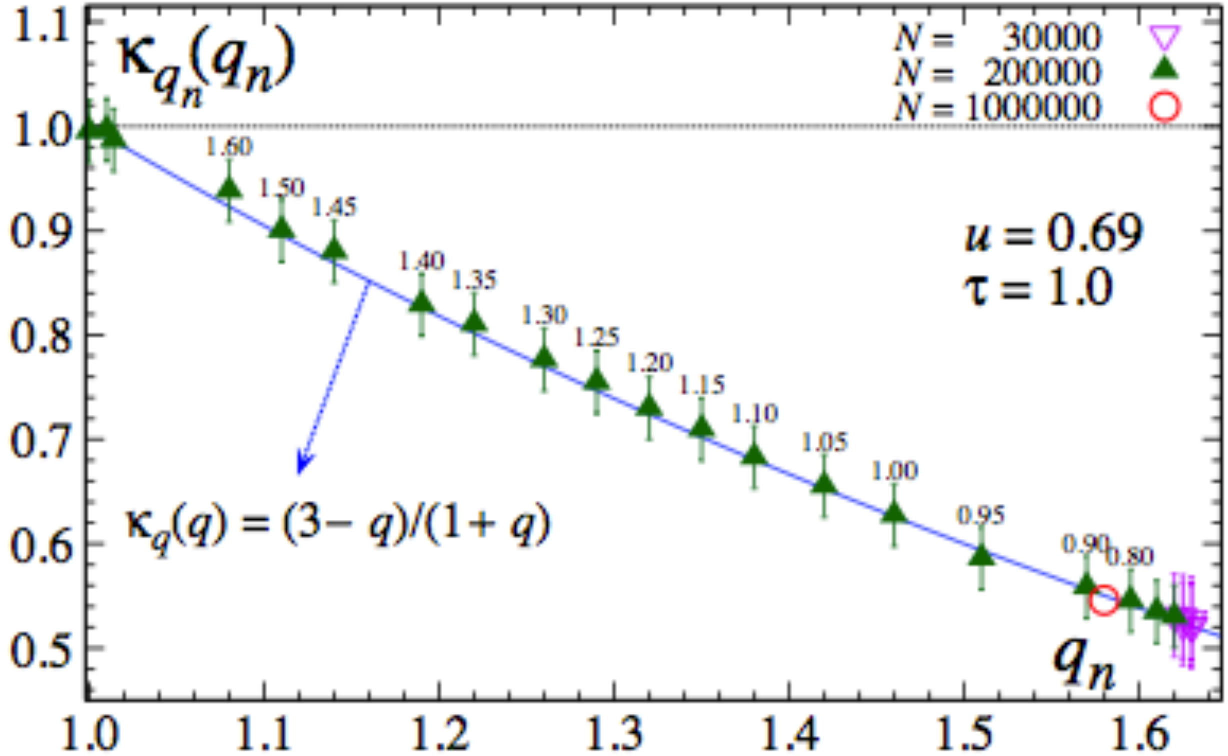
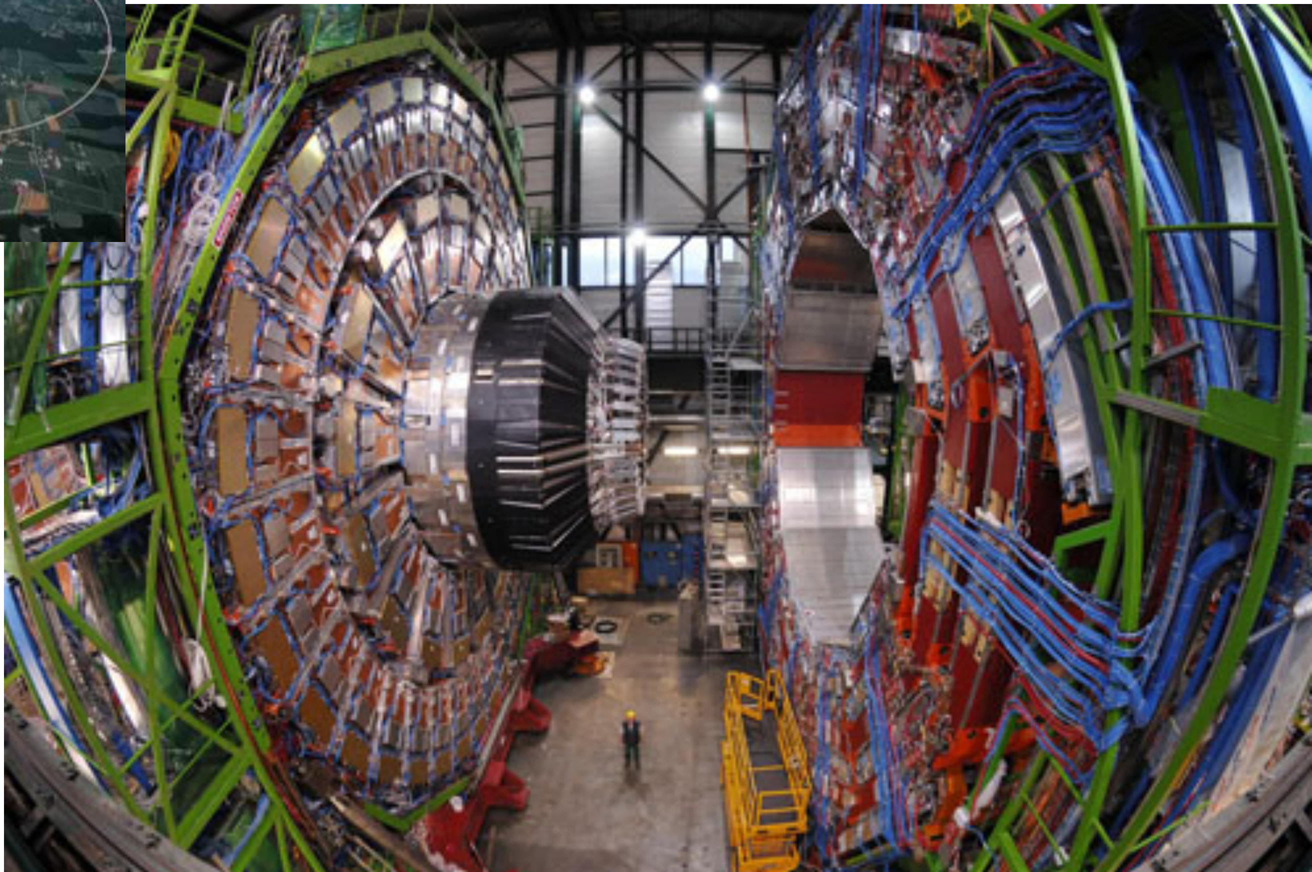


FIG. 3: q_n and q-kurtosis κ_{q_n} that have been obtained from the histograms corresponding to typical values of α (numbers indicated on top of the points). The red circle corresponds to Fig. 2. The continuous curve $\kappa_q = (3 - q)/(1 + q)$ is the analytical one obtained with q-Gaussians. Notice that κ_q is finite up to $q = 3$ (maximal admissible value for a q-Gaussian to be normalizable), and that it does not depend on β_{q_n}. The visible departure from the dotted line at $\kappa_q = 1$ corresponding to a Maxwellian distribution, neatly reflects the departure from BG thermostatistics.

LHC (Large Hadron Collider)

CMS (Compact Muon Solenoid) detector

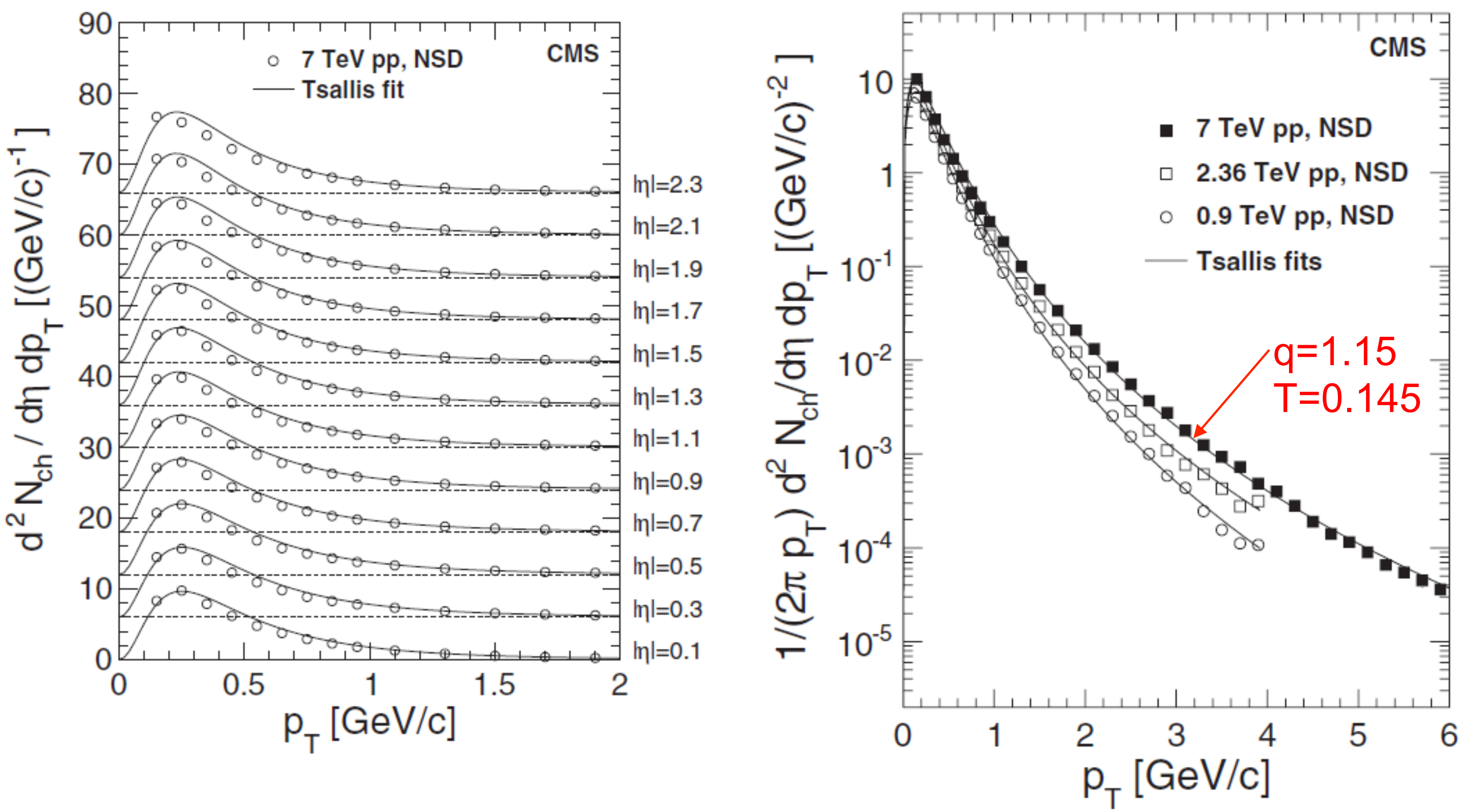
~ 2500 scientists/engineers from 183 institutions of 38 countries



Transverse-Momentum and Pseudorapidity Distributions of Charged Hadrons in pp Collisions at $\sqrt{s} = 7$ TeV

V. Khachatryan *et al.**
(CMS Collaboration)

(Received 18 May 2010; published 6 July 2010)



Eur. Phys. J. C (2011) 71:1655
DOI 10.1140/epjc/s10052-011-1655-9

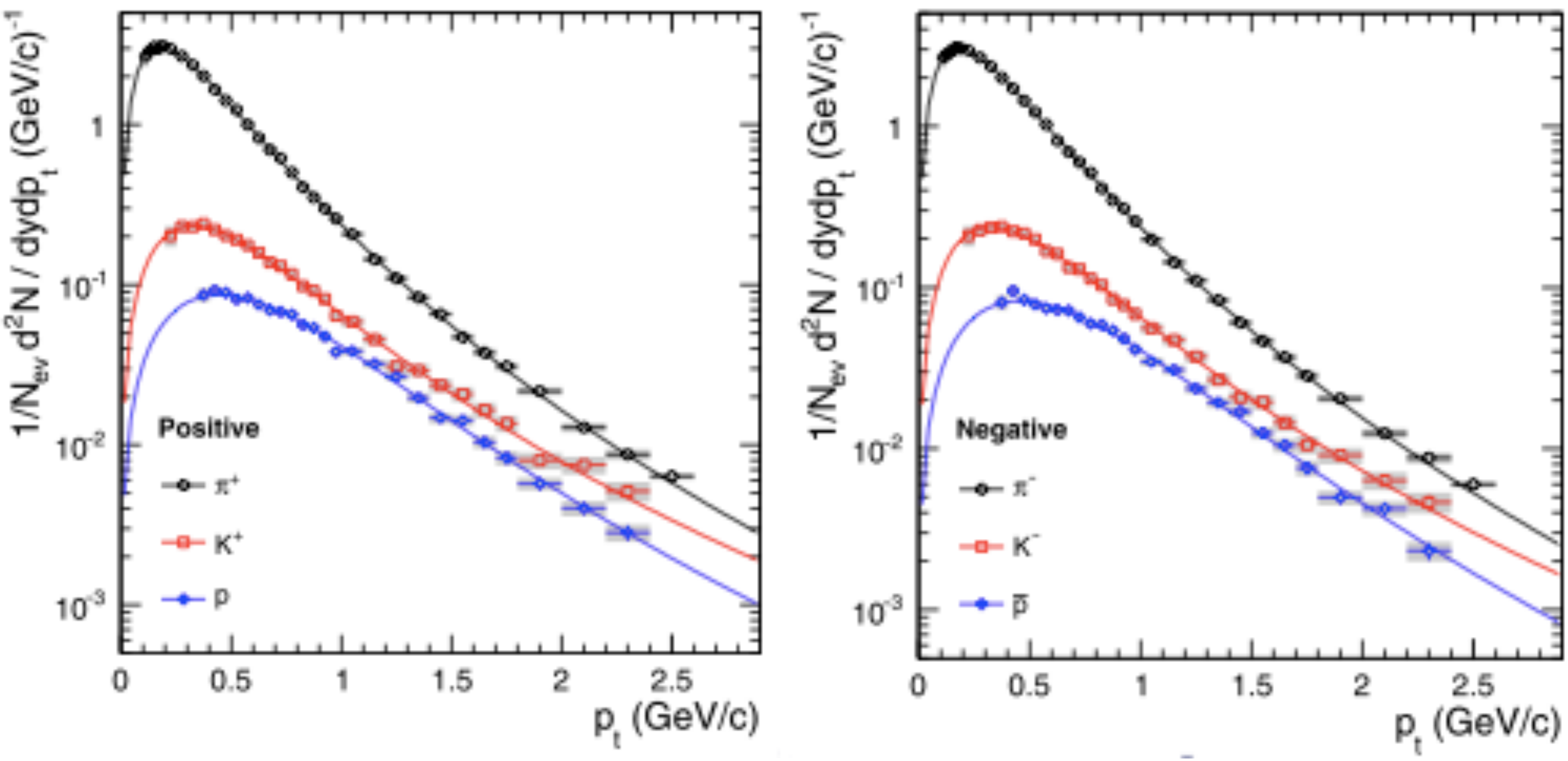
THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article - Experimental Physics

Production of pions, kaons and protons in pp collisions at $\sqrt{s} = 900$ GeV with ALICE at the LHC

The ALICE Collaboration

K. Aamodt[77], N. Abel[43], U. Abeysekara[75], A. Abrahantes Quintana[42], A. Abramyan[112], D. Adamová[85], M.M. Aggarwal[25], G. Aglieri Rinella[40], A.G. Agocs[18], S. Aguilar Salazar[63], Z. Ahammed[53], A. Ahmad[2], N. Ahmad[2], S.U. Ahn[38,b], R. Akimoto[99], A. Akindinov[66], D. Aleksandrov[68], B. Alessandro[104], R. Alfaro Molina[63], A. Alici[13], E. Almaráz Aviña[63], J. Alme[8], T. Alt[43,c], V. Altini[5], S. Altinpinar[31], C. Andrei[17], A. Andronic[31], G. Anelli[40], V. Angelov[43,c], C. Anson[27], T. Antičić[113], F. Antinori[40,d], S. Antinori[13], K. Antipin[36], D. Antończyk[36], P. Antonioli[14], A. Anzo[63], L. Aphecetche[71], H. Appelshäuser[36], S. Arcelli[13], R. Arceo[63], A. Arend[36], N. Armesto[91], R. Arnaldi[104], T. Aronsson[72], I.C. Arsene[77,e], A. Asryan[97], A. Augustinus[40], R. Averbeck[31], T.C. Awes[74], J. Äystö[49], M.D. Azmi[2], S. Bablok[8], M. Bach[35], A. Badalà[24], Y.W. Baek[38,b], S. Bagnasco[104], R. Bailhache[31,f], R. Bala[105], A. Baldisseri[88], A. Baldit[26], J. Bán[56], R. Barbera[23], G.G. Barnaföldi[18], L.S. Barnby[12], V. Barret[26], J. Bartke[29], F. Barile[5], M. Basile[13], V. Basmanov[93], N. Bastid[26], B. Bathen[70], G. Batigne[71], B. Batyunya[34], C. Baumann[70,f], I.G. Bearden[28], B. Becker[20,g], I. Belikov[98], R. Bellwied[33], E. Belmont-Moreno[63], A. Belogianni[4], L. Benhabib[71], S. Beole[103], I. Berceanu[17], A. Bercuci[31,h], E. Berdermann[31], Y. Berdnikov[39], L. Betev[40], A. Bhasin[48], A.K. Bhati[25], L. Bianchi[103], N. Bianchi[37], C. Bianchin[78], J. Bielčík[80], J. Bielčíková[85], A. Bilandzic[3], L. Bimbot[76], E. Biolcati[103], A. Blanc[26], F. Blanco[23,i], F. Blanco[61], D. Blau[68], C. Blume[36], M. Boccioli[40], N. Bock[27], A. Bogdanov[67], H. Bøggild[28], M. Bogolyubsky[82], J. Bohm[95], L. Boldizsár[18], M. Bombara[55], C. Bombonati[78,k], M. Bondila[49], H. Borel[88], A. Borisov[50], C. Bortolin[78,ao], S. Bose[52], L. Bosisio[100], F. Bossú[103], M. Botje[3], S. Böttger[43], G. Bourdaud[71], B. Boyer[76], M. Braun[97], P. Braun-Munzinger[31,32,c], L. Bravina[77], M. Bregant[100,l], T. Breitner[43], G. Bruckner[40], R. Brun[40], E. Bruna[72], G.E. Bruno[5], D. Budnikov[93], H. Buesching[36], P. Buncic[40], O. Busch[44], Z. Buthelezi[22], D. Caffarri[78], Y. Cai[111], H. Caines[72], E. Calvo[58], E. Camacho[64], P. Camerini[100], M. Campbell[40], V. Canoa Roman[40]



1/N_ev d²N / dydp_t (GeV/c)^-1
p_t (GeV/c)
Positive
π⁺
K⁺
p
Negative
π⁻
K⁻
p̄

New Journal of Physics
The open-access journal for physics

Charged-particle multiplicities in *pp* interactions measured with the ATLAS detector at the LHC

The ATLAS Collaboration

New Journal of Physics **13** (2011) 053033 (68pp)
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Online at http://www.njp.org/
doi:10.1088/1367-2630/13/5/053033

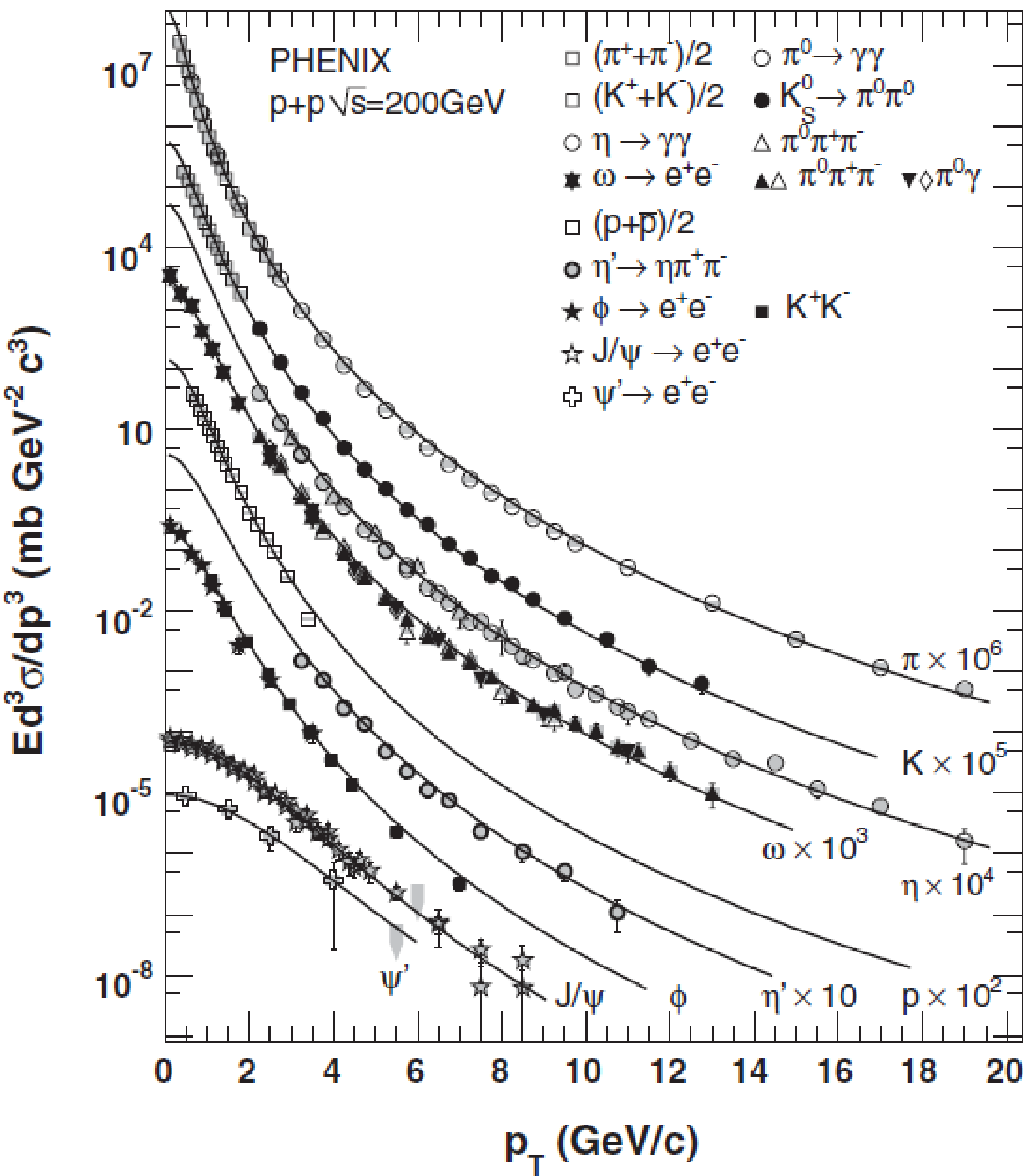
Abstract. Measurements are presented from proton–proton collisions at centre-of-mass energies of $\sqrt{s} = 0.9$, 2.36 and 7 TeV recorded with the ATLAS detector at the LHC. Events were collected using a single-arm minimum-bias trigger. The charged-particle multiplicity, its dependence on transverse momentum and pseudorapidity and the relationship between the mean transverse momentum and charged-particle multiplicity are measured. Measurements in different regions of phase space are shown, providing diffraction-reduced measurements as well as more inclusive ones. The observed distributions are corrected to well-defined phase-space regions, using model-independent corrections. The results are compared to each other and to various Monte Carlo (MC) models, including a new AMBT1 PYTHIA6 tune. In all the kinematic regions considered, the particle multiplicities are higher than predicted by the MC models. The central charged-particle multiplicity per event and unit of pseudorapidity, for tracks with $p_T > 100$ MeV, is measured to be 3.483 ± 0.009 (stat) ± 0.106 (syst) at $\sqrt{s} = 0.9$ TeV and 5.630 ± 0.003 (stat) ± 0.169 (syst) at $\sqrt{s} = 7$ TeV.

PHYSICAL REVIEW D **83,** 052004 (2011)

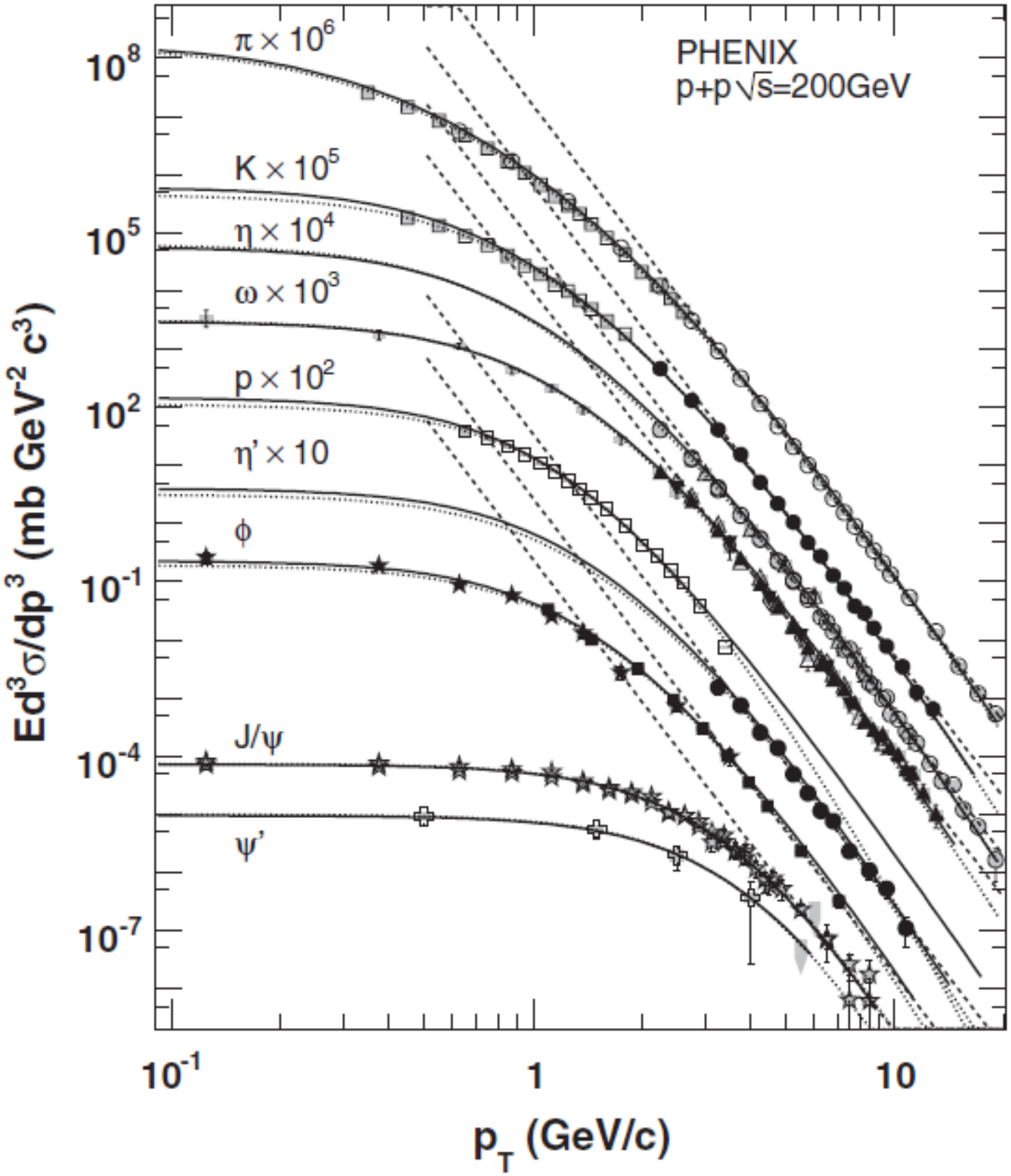
Measurement of neutral mesons in $p + p$ collisions at $\sqrt{s} = 200$ GeV and scaling properties of hadron production

A. Adare,[11] S. Afanasiev,[25] C. Aidala,[12,36] N. N. Ajitanand,[53] Y. Akiba,[47,48] H. Al-Bataineh,[42] J. Alexander,[53] K. Aoki,[30,47] L. Aphecetche,[55] R. Armendariz,[42] S. H. Aronson,[6] J. Asai,[47,48] E. T. Atomssa,[31] R. Averbeck,[54] T. C. Awes,[43] B. Azmoun,[6] V. Babintsev,[21] M. Bai,[5] G. Baksay,[17] L. Baksay,[17] A. Baldisseri,[14] K. N. Barish,[7] P. D. Barnes,[33] B. Bassalleck,[41] A. T. Basye,[1] S. Bathe,[7] S. Batsouli,[43] V. Baublis,[46] C. Baumann,[37] A. Bazilevsky,[6] S. Belikov,[6,*] R. Bennett,[54] A. Berdnikov,[50] Y. Berdnikov,[50] A. A. Bickley,[11] J. G. Boissevain,[33] H. Borel,[14] K. Boyle,[54] M. L. Brooks,[33] H. Buesching,[6] V. Bumazhnov,[21] G. Bunce,[6,48] S. Butsyk,[33,54] C. M. Camacho,[33] S. Campbell,[54] B. S. Chang,[62] W. C. Chang,[2] J.-L. Charvet,[14] S. Chernichenko,[21] J. Chiba,[26] C. Y. Chi,[12] M. Chiu,[22] I. J. Choi,[62] R. K. Choudhury,[4] T. Chujo,[58,59] P. Chung,[53] A. Churyn,[21] V. Cianciolo,[43] Z. Citron,[54] C. R. Cleven,[19] B. A. Cole,[12] M. P. Comets,[44] P. Constantin,[33] M. Csanád,[16] T. Csörgő,[27] T. Dahms,[54] S. Dairaku,[30,47] K. Das,[18] G. David,[6] M. B. Deaton,[1] K. Dehmelt,[17] H. Delagrange,[55] A. Denisov,[21] D. d'Enterria,[12,31] A. Deshpande,[48,54] E. J. Desmond,[6] O. Dietzsch,[51] A. Dion,[54] M. Donadelli,[51] O. Drapier,[31] A. Drees,[54] K. A. Drees,[5] A. K. Dubey,[61] A. Durum,[21] D. Dutta,[4] V. Dzhordzhadze,[7] Y. V. Efremenko,[43] J. Egdemir,[54] F. Ellinghaus,[11] W. S. Emam,[7] T. Engelmore,[12] A. Enokizono,[32] H. En'yo,[47,48] S. Esumi,[58] K. O. Eyser,[7] B. Fadem,[38] D. E. Fields,[41,48] M. Finger, Jr.,[8,25] M. Finger,[8,25] F. Fleuret,[31] S. L. Fokin,[29] Z. Fraenkel,[61,*] J. E. Frantz,[54] A. Franz,[6] A. D. Frawley,[18] K. Fujiwara,[47] Y. Fukao,[30,47] T. Fusayasu,[40] S. Gadrat,[34] I. Garishvili,[56] A. Glenn,[11] H. Gong,[54] M. Gonin,[31] J. Gosset,[14] Y. Goto,[47,48] R. Granier de Cassagnac,[31] N. Grau,[12,24] S. V. Greene,[59] M. Grosse Perdekamp,[22,48] T. Gunji,[10] H. -Å. Gustafsson,[35,*] T. Hachiya,[20] A. Hadj Henni,[55] C. Haegemann,[41] J. S. Haggerty,[6] H. Hamagaki,[10] R. Han,[45] H. Harada,[20] E. P. Hartouni,[32] K. Haruna,[20] E. Haslum,[35] R. Hayano,[10] M. Heffner,[32] T. K. Hemmick,[54] T. Hester,[7] X. He,[19] H. Hiejima,[22] J. C. Hill,[24] R. Hobbs,[41] M. Hohlmann,[17] W. Holzmann,[53] K. Homma,[20] B. Hong,[28] T. Horaguchi,[10,47,57] D. Hornback,[56] S. Huang,[59] T. Ichihara,[47,48] R. Ichimiya,[47] H. Iinuma,[30,47] Y. Ikeda,[58] K. Imai,[30,47] J. Imrek,[15] M. Inaba,[58] Y. Inoue,[49,47] D. Isenhower,[1] L. Isenhower,[1] M. Ishihara,[47] T. Isobe,[10] M. Issah,[53] A. Isupov,[25] D. Ivanischev,[46] B. V. Jacak,[54,†] J. Jia,[12] J. Jin,[12] O. Jinnouchi,[48] B. M. Johnson,[6] K. S. Joo,[39] D. Jouan,[44] F. Kajihara,[10] S. Kametani,[10,47,60] N. Kamihara,[47,48] J. Kamin,[54] M. Kaneta,[48] J. H. Kang,[62] H. Kanou,[47,57] J. Kapustinsky,[33] D. Kawall,[36,48] A. V. Kazantsev,[29] T. Kempel,[24] A. Khanzadeev,[46] K. M. Kijima,[20] J. Kikuchi,[60] B. I. Kim,[28] D. H. Kim,[39] D. J. Kim,[62] E. Kim,[52] S. H. Kim,[62] E. Kinney,[11] K. Kiriluk,[11] Á. Kiss,[16] E. Kistenev,[6] A. Kiyomichi,[47] J. Klay,[32] C. Klein-Boesing,[37] L. Kochenda,[46] V. Kochetkov,[21] B. Komkov,[46] M. Konno,[58] J. Koster,[22] D. Kotchetkov,[7] A. Kozlov,[61] A. Král,[13] A. Kravitz,[12] J. Kubart,[8,23] G. J. Kunde,[33] N. Kurihara,[10]

PHENIX @ RHIC


PHENIX
p+p √s=200GeV
(π⁺+π⁻)/2
(K⁺+K⁻)/2
η → γγ
ω → e⁺e⁻
(p+p̄)/2
η’→ ηπ⁺π⁻
φ → e⁺e⁻
J/ψ → e⁺e⁻
ψ’→ e⁺e⁻
π⁰→ γγ
K⁰S→ π⁰π⁰
π⁰π⁺π⁻
π⁰π⁺π⁻
π⁰γ
K⁺K⁻
π × 10⁶
K × 10⁵
ω × 10³
η × 10⁴
ψ’
J/ψ
φ
η’ × 10
p × 10²
Ed³σ/dp³ (mb GeV⁻² c³)
p_T (GeV/c)

$q \simeq 1.10$



$q \simeq 1.10$

FIG. 13. The p_T spectra of various hadrons measured by PHENIX fitted to the power law fit (dashed lines) and Tsallis fit (solid lines). See text for more details.

Nonlinear Relativistic and Quantum Equations with a Common Type of Solution

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Generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, are proposed. Nonlinear terms, characterized by exponents depending on an index q, are considered in such a way that the standard linear equations are recovered in the limit $q \to 1$. Interestingly, these equations present a common, solitonlike, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. In all cases, the well-known Einstein energy-momentum relation is preserved for arbitrary values of q.

$q-$ generalized Schroedinger equation

(quantum non-relativistic spinless free particle)

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi\left(\vec{x},t\right)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Phi\left(\vec{x},t\right)}{\Phi_0}\right]^{2-q} \qquad (q \in R)$$

Its exact solution is given by

$$\Phi\left(\vec{x},t\right) = \Phi_0\; e_q^{i\left(\vec{p}\,.\,\vec{x}-Et\right)/\hbar} = \Phi_0\; e_q^{i\left(\vec{k}\,.\,\vec{x}-\omega t\right)}$$

with

$$\left.\begin{aligned} &E = \frac{p^2}{2m} \quad \text{(Newtonian relation!)} \\ &E = \hbar\omega \\ &p = \hbar k \end{aligned}\right\} \forall q$$

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q-generalized Klein-Gordon equation:

(quantum relativistic spinless free particle: e.g., mesons π)

$$\nabla^2 \Phi(\vec{x},t) = \frac{1}{c^2}\frac{\partial^2 \Phi(\vec{x},t)}{\partial t^2} + q\frac{m^2c^2}{\hbar^2}\Phi(\vec{x},t)\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2(q-1)} \qquad (q \in R)$$

Its exact solution is given by

$$\Phi(\vec{x},t) = \Phi_0 \; e_q^{i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \Phi_0 \; e_q^{i(\vec{k}\,.\,\vec{x}-\omega\, t)}$$

with

$$E^2 = p^2c^2 + m^2c^4 \qquad (\forall q) \qquad \text{(Einstein relation!)}$$

Particular case: $\quad m = 0 \;\Rightarrow\; q$-plane waves

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q-generalized Dirac equation:

(quantum relativistic spin $1/2$ matter and anti-matter free particles:

e.g., electron and positron)

$$i\hbar \frac{\partial\, \Phi(\vec{x},t)}{\partial t} + i\hbar c\left(\vec{\alpha}.\vec{\nabla}\right)\Phi(\vec{x},t) = \beta m c^2 A^{(q)}(\vec{x},t)\;\Phi(\vec{x},t) \qquad (q \in R)$$

with

$$\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \quad \beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix} \qquad (4 \times 4 \text{ matrices})$$

$$A^{(q)}_{i\,j}(\vec{x},t) \equiv \delta_{ij}\left[\frac{\Phi_j(\vec{x},t)}{a_j}\right]^{q-1} \qquad \left(A^{(1)}_{i\,j}(\vec{x},t) = \delta_{ij}\right) \qquad (4 \times 4 \text{ matrix})$$

where $\{a_j\}$ are complex constants.

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Its exact solution is given by

$$\Phi(\vec{x},t) \equiv \begin{pmatrix} \Phi_1(\vec{x},t) \\ \Phi_2(\vec{x},t) \\ \Phi_3(\vec{x},t) \\ \Phi_4(\vec{x},t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{k}\,.\,\vec{x}-\omega\, t)}$$

with $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ being the same $\forall q$

hence

$$E^2 = p^2c^2 + m^2c^4 \quad (q \in R) \quad \text{(Einstein relation!)}$$

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BOOKS AND SPECIAL ISSUES ON NONEXTENSIVE STATISTICAL MECHANICS

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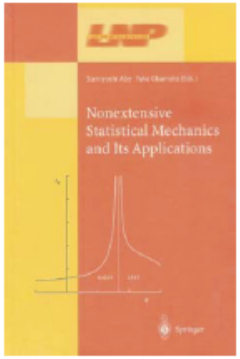

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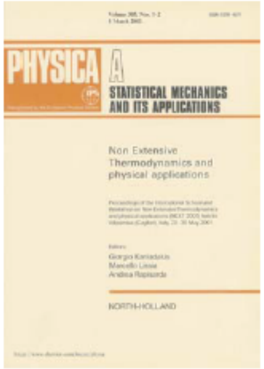

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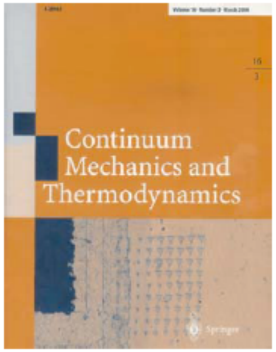

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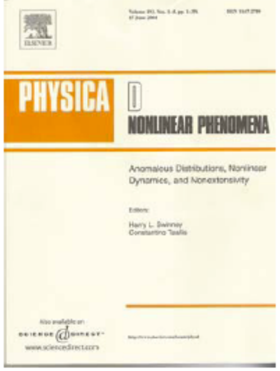

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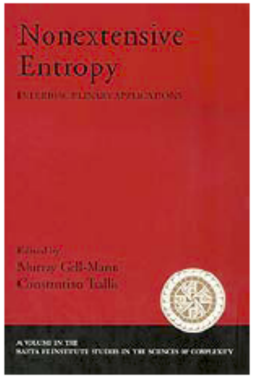

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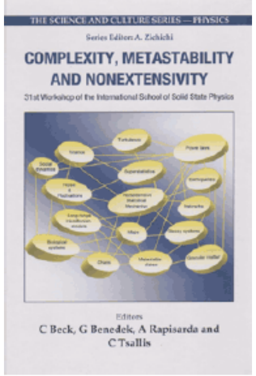

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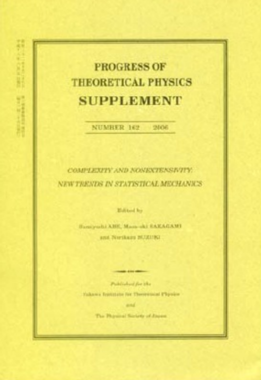

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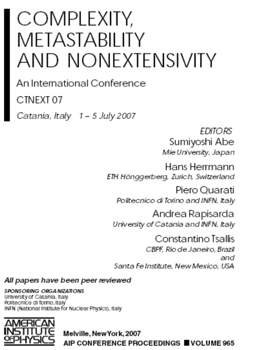

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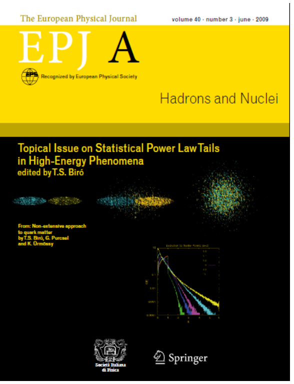

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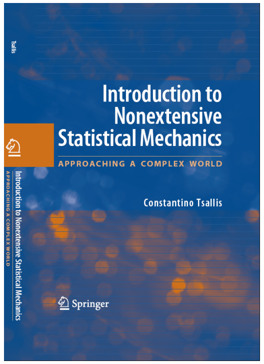

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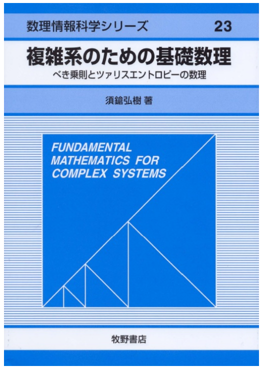

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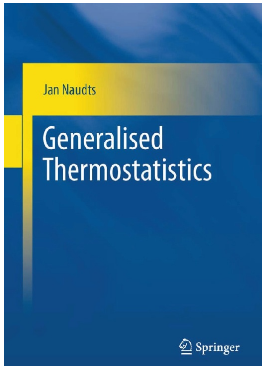

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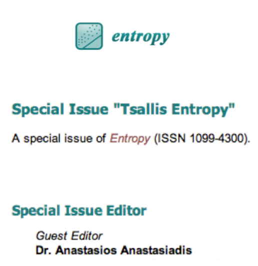

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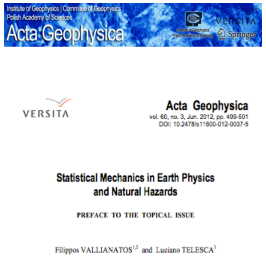

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Introduction to Nonextensive Statistical Mechanics

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